

LIFTING HOMOTOPY T -ALGEBRA MAPS TO STRICT MAPS

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ABSTRACT. The settings for homotopical algebra—categories such as simplicial groups, simplicial rings, A_∞ spaces, E_∞ ring spectra, etc.—are oftentimes equivalent to categories of algebras over some monad or triple T . In such cases, T is acting on a nice simplicial model category in such a way that the T descends to a monad on the homotopy category and defines a category of *homotopy* T -algebras. In this setting there is a forgetful functor from the homotopy category of T -algebras to the category of homotopy T -algebras.

Under suitable hypotheses we provide an obstruction theory, in the form of a Bousfield-Kan spectral sequence, for lifting a homotopy T -algebra map to a strict map of T -algebras. Once we have a map of T -algebras to serve as a basepoint, the spectral sequence computes the homotopy groups of the space of T -algebra maps and the edge homomorphism on π_0 is the aforementioned forgetful functor. We discuss a variety of settings in which the required hypotheses are satisfied, including monads arising from algebraic theories and from operads.

We provide examples in G -spaces, G -spectra, rational E_∞ -algebras, and A_∞ -algebras under an Eilenberg-MacLane commutative ring spectrum. We give explicit calculations, connected to rational unstable homotopy theory, showing that the forgetful functor from the homotopy category of E_∞ ring spectra to the category of H_∞ ring spectra is generally neither full nor faithful. We also apply a result of the second named author and Nick Kuhn to compute the homotopy type of the space $E_\infty(\Sigma_+^\infty \operatorname{Coker} J, L_{K(2)} R)$.

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Date: February 25, 2013.

2010 *Mathematics Subject Classification.* Primary: 55P99, 55S35; Secondary: 13D03, 18C15, 18C10, 18G55, 55P43, 55P62, 55T05, 55Q50.

1. INTRODUCTION

In the work of Ando, Hopkins, Rezk, and Strickland on the Witten genus [AHS01, AHS04, AHR06] the authors first construct a lift of the Witten genus to a multiplicative map of cohomology theories, then to an H_∞ map (i.e., a map preserving power operations), and finally to an E_∞ map

$$MString \rightarrow tmf.$$

In each of these steps they are asking that the map respects additional structure and it is natural to ask if there are general techniques for constructing such liftings.

Their construction of an H_∞ map makes use of ideas from Ando's thesis [And92, And95], where he defines H_∞ maps from complex cobordism to Lubin-Tate spectra using a connection to isogenies of Lubin-Tate formal group laws. Thus the result arises from a computation: Since the H_∞ condition can be formulated in the stable homotopy category, a map is H_∞ if and only if an associated sequence of cohomological equations hold. The applicability of such techniques is one of the reasons that the category of H_∞ ring spectra is computationally more accessible. Although every E_∞ map forgets to an H_∞ map, constructing E_∞ maps is much more subtle and requires rather different techniques.

We construct an obstruction-theoretic spectral sequence to detect when an H_∞ map can be lifted to an E_∞ map and many other problems of this type. As a consequence of our approach we can also see *how much* information is lost under the passage from E_∞ to H_∞ ring spectra. The first category can be described as the category of algebras over a monad/triple T in a category of spectra while the second is the category of such algebras in the homotopy category. Phrased in these terms, it is expected that a great deal is forgotten in the passage from E_∞ to H_∞ ring spectra. But to date, there have been no examples demonstrating this. Since our methods apply more generally to studying categories of algebras over a monad T (satisfying some hypotheses), we set up our machinery in the more abstract setting.

In Section 2 we provide a rapid review of the theory of monads and how they naturally encode algebraic structures. We emphasize the examples coming from algebraic theories and from operads since they make up the majority of our examples. In Section 3.1, we recall some conditions which guarantee the existence of a simplicial model structure on the category of algebras over a monad. These conditions are often satisfied and cover a broad range of standard examples. We include this standard material so the reader can easily apply it to the application of their choosing.

Our first main result is:

Theorem 1.1. Suppose T is a monad acting on a simplicial category \mathcal{C} and X and Y are T -algebras such that:

- (a) T is Quillen (Definition 3.1),
- (b) T commutes with geometric realization,
- (c) and X is resolvable with replacement \tilde{X} (Definition 3.18).

Let $U: \mathcal{C}_T \rightarrow \mathcal{C}$ denote the forgetful functor from the category of T -algebras to \mathcal{C} . Then T induces a monad hT on $ho\mathcal{C}$ and there exists an obstruction-theoretic spectral sequence, called the T -algebra spectral sequence, satisfying:

- (1) Provided a T -algebra map $\varepsilon: X \rightarrow Y$ to serve as a base point, the spectral sequence conditionally converges to the homotopy of the derived mapping space

$$\pi^s \pi_t \mathcal{C}^d(T^\bullet U \tilde{X}, UY) \Rightarrow \pi_* \mathcal{C}_T^d(X, Y).$$

- (2) In this case the differentials $d_r[f]$ provide obstructions to lifting $[f]$ to a map of T -algebras.

(3) The edge homomorphisms

$$\begin{aligned}
\pi_0 \mathcal{C}_T^d(X, Y) &\twoheadrightarrow E_\infty^{0,0} \\
&\hookrightarrow E_2^{0,0} = (ho\mathcal{C})_{hT}(UX, UY) \\
&\hookrightarrow E_1^{0,0} = ho\mathcal{C}(UX, UY)
\end{aligned}$$

are the corresponding forgetful functors.

- (4) The spectral sequence is contravariantly functorial in $X \in ho(\mathcal{C}_T)$ and covariantly functorial in $Y \in ho(\mathcal{C}_T)$ and T satisfying the hypotheses.

This result will be proven in Section 4.1. Note that we do not require any properness assumptions on our model category, since we avoid using E_2 model structures or Bousfield localizations. The Quillen assumption on our monad is innocuous and satisfied in practice. The remaining two assumptions guarantee convergence to the desired target and allow us to identify the key terms in the spectral sequence.

Note that the convergence result in Theorem 1.1 is stronger than that of alternative approaches found in the literature, e.g. using a Reedy cofibrant replacement of the bar resolution or taking homotopy colimits instead of geometric realization. These approaches give spectral sequences which converge to mapping spaces from a T -cocompletion of the source, as in [Hes10, BR12b]. We combine some standard results recalled in Section 3.2 with some crucial new technical lemmas in Section 3.3 to prove convergence without a cocompletion under the additional assumptions of Theorem 1.1.

In Section 4.3 we show that these assumptions hold in many general cases of interest such as nice categories of algebras over an operad, G -spaces and G -spectra (provided G is sufficiently nice), and many algebraic categories such as simplicial groups and rings.

Bousfield has shown that his spectral sequence can still be applied even without the existence of a base point—a useful generalization since a space of T -algebra maps may well be empty. In this case there is an obstruction theory (see Remark 4.4) for lifting a map in \mathcal{C} to a map of T -algebras so that one can obtain a base point [Bou89, §5]. The farther one can lift this base point up the totalization tower, the greater the range in which one can define the spectral sequence and differentials.

As shown in Theorem 4.5, when the relevant mapping spaces in \mathcal{C} have the homotopy type of H -spaces, e.g., if $\mathcal{C} = Spectra$, then one can choose these obstructions to land in the E_2 page of the spectral sequence. Under favorable circumstances we can apply our second main theorem, Theorem 5.6, to identify the E_2 term with André-Quillen cohomology groups. This is a significant computational tool.

We then demonstrate the wide applicability of this spectral sequence and its computability through a number of examples in Section 5. The reader interested in applications is encouraged to skip directly to this section where we compute the homotopy groups of:

- Spaces of equivariant maps in G -spaces and G -spectra (Section 5.1). This is a warm-up for the other examples. We explicitly analyze the forgetful functor from the homotopy category of (strict) G -objects to (weak) G -objects in the homotopy category of spaces or spectra.
- Spaces of E_∞ maps between function spectra (Section 5.3). In two examples from unstable rational homotopy theory, we show that the forgetful functor from E_∞ to H_∞ ring spectra is generally neither full nor faithful. To the authors' knowledge, these are the first such examples.
- Spaces of A_∞ and E_∞ self-maps of particular Hk -algebras, for k a suitable field, whose homotopy rings are polynomial algebras.

- Spaces of E_∞ maps from $\Sigma_+^\infty \text{Coker } J$ to a $K(2)$ -local E_∞ ring spectrum (Section 5.4). This is a result of Nick Kuhn and the second named author, and gives a nontrivial example of when the set of H_∞ maps coincides with the set of homotopy classes of E_∞ maps. As a consequence of the proof we obtain new information about $\text{Coker } J$, including its $K(2)$ -homology:

$$K(2)_* \text{Coker } J \cong \bigoplus_{n \geq 0} K(2)_* B\Sigma_n.$$

Several of these examples make use of our second main result which gives general conditions for identifying the E_2 term of the T -algebra spectral sequence with André-Quillen cohomology.

Related work. The T -algebra spectral sequence arises by taking a functorial resolution of the source X . Namely we replace X by the two sided bar construction $B(F_T, T, UX)$ where U is the forgetful functor $\mathcal{C}_T \rightarrow \mathcal{C}$ and F_T is its left adjoint. For this approach, one wants general conditions under which the replacement is cofibrant, weakly equivalent to X , and equipped with a suitable filtration for obtaining a spectral sequence. A number of special cases of this theory are well known, and the arguments for spaces and spectra can be found in the literature. Although the two-sided bar construction has been a standard tool in homotopy theory for decades, we know of no reference in which its homotopical properties are developed with sufficient breadth for our purposes. In Section 3.3 we develop a new tool for this purpose and apply it in Section 4.3 to demonstrate the applicability of our spectral sequence.

There are a couple of alternative methods for constructing maps of structured ring spectra. This work can be considered an extension of the obstruction theory for maps of A_∞ simplicial R -modules and A_∞ ring spectra that appears in Rezk's thesis [Rez96] and his presentation of the Hopkins-Miller theorem [Rez97]. Indeed the latter work was a significant source of inspiration for this project. Angeltveit [Ang08] has also constructed an obstruction theory, which appears to be part of a spectral sequence, for computing maps of A_∞ ring spectra.

The Goerss-Hopkins spectral sequence also computes the homotopy of the derived mapping space between two spectra which are algebras over a suitable operad, such as an E_∞ operad [GH04, GH05]. This spectral sequence uses an E_2 model structure which guarantees an algebraic description of the E_2 term and is, in general, *not* the same as the T -algebra spectral sequence. In particular, their edge homomorphism is generally a Hurewicz homomorphism which generally is distinct from the forgetful functor above. However, in future work, the second named author will show that in special cases such as those considered in Section 5.3 the spectral sequences do agree and computations can be done in either framework.

When Theorem 5.6 does not apply, it is generally quite difficult to determine the E_2 term of the T -algebra spectral sequence. Indeed, when T is the monad associated to the E_∞ operad, the main results of [AHS04, And95, JN10] could be expressed as partial computations of $d_1: E_1^{0,0} \rightarrow E_1^{1,0}$. The difficulties here are generic: there are very few examples where one has enough knowledge of the power operations to compute the E_2 term explicitly.

Acknowledgements. This work began when the authors were visiting the Max Planck Institute for Mathematics. The authors would like to thank the MPIM for its hospitality during their stay. The second named author would also like to thank Benoit Fress, Paul Goerss, Stefan Schwede, Karol Szumilo, and Markus Szymik for helpful conversations concerning the material below. The authors also gratefully acknowledge partial support from the University of Bonn and the Deutsche Forschungsgemeinschaft through Graduiertenkolleg 1150, as well as the University of Georgia through VIGRE II.

Conventions/Terminology. We will make the convention that a simplicial category is a simplicially enriched category which is tensored and cotensored over simplicial sets. This convention is standard when discussing simplicial model categories, but unusual in enriched category theory.

2. ALGEBRAS OVER A MONAD

This section reviews monads and their categories of algebras, focusing on examples and conditions which ensure that limits and colimits in the categories of algebras exist. The existence of these constructions is not automatic, but will be essential for the material in Section 3. We also show how these constructions are computed in practice.

In Section 2.1 we begin with a familiar example, focusing on points which are key to the general theory. A wealth of additional examples can be found in the framework of algebraic theories which we recall in Section 2.2. In Section 2.3 we extend this discussion to the simplicially enriched context. Finally we recall some relevant facts about operads in Section 2.4. In these last two sections we introduce two of our primary classes of examples: Simplicial algebraic theories and operads.

2.1. Monadicity and categories of algebras. Given a set S we can form the free group FS on S whose underlying set consists of all finite reduced words whose letters are signed elements of S . Multiplication is then defined by composing words. We can also take a group G , forget its group structure, and regard it as a set $X = UG$. These constructions are clearly functorial and participate in an adjunction

$$\mathcal{G}roup \begin{matrix} \xrightarrow{U} \\ \xleftarrow{F} \end{matrix} \mathcal{S}et$$

where U is right adjoint to F . Let $T = UF$ denote the endofunctor of $\mathcal{S}et$ given by the composite of these two functors.

The unit of this adjunction is a natural transformation $e: \text{Id} \rightarrow T$ given by taking an element of a set to its associated word of length one. Using the underlying group structure on $X = UG$ one can multiply the elements in a word to obtain a *structure map*

$$\mu_X: TX \rightarrow X.$$

Alternatively we could construct this map by applying U to the counit

$$\varepsilon: FU \rightarrow \text{Id}$$

of this adjunction. In particular, we have such a map for anything in the image of T and obtain a natural transformation

$$\mu_T: T^2 \rightarrow T.$$

The (large) category of endofunctors of $\mathcal{S}et$ admits a monoidal structure under composition and we can see that (T, e, μ_T) is an associative monoid in this category, in other words, T is a *monad* on $\mathcal{S}et$.

In the case of $X = UG$ we see that the map μ_X is compatible with this structure in the sense that the two double composites of straight arrows in (2.1) are equal and each composite of a curved arrow followed by a straight arrow is the identity morphism.

$$(2.1) \quad \begin{array}{ccccc} & \overset{e_{TX}}{\curvearrowright} & & \overset{e_X}{\curvearrowright} & \\ & \mu_T & & \mu_X & \\ TTX & \xrightarrow{\quad} & TX & \xrightarrow{\quad} & X \\ & \underset{T\mu_X}{\xleftarrow{\quad}} & & & \end{array}$$

An object $X \in \mathcal{S}et$ with a map $\mu_X: TX \rightarrow X$ satisfying these identities is called a T -algebra in $\mathcal{S}et$. We obtain a category $\mathcal{S}et_T$ of T -algebras in $\mathcal{S}et$ by restricting to those set maps which commute with the structure morphisms. To be explicit, the morphisms between two T -algebras (X, μ_X) and

(Y, μ_Y) are those maps $f : X \rightarrow Y$ such that the following diagram commutes:

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ \mu_X \downarrow & & \downarrow \mu_Y \\ X & \xrightarrow{f} & Y \end{array}$$

or, alternatively,

$$(2.2) \quad \text{Set}_T(X, Y) = \text{eq} \left[\text{Set}(X, Y) \xrightarrow[\mu_X^*]{(\mu_Y \circ Tf)^*} \text{Set}(TX, Y) \right].$$

The category of T -algebras in Set admits an obvious forgetful functor to Set and we saw above that the forgetful functor $U : \text{Group} \rightarrow \text{Set}$ factors through Set_T . It is not difficult to see that the latter functor defines an equivalence of categories $\text{Group} \simeq \text{Set}_T$. Indeed, if G is a group then we can see that some of the maps in (2.1) can be realized by applying U to following diagram of groups:

$$(2.3) \quad \begin{array}{ccc} & \xleftarrow{e} & \\ & \mu_T & \\ FTUG & \xrightarrow[F\mu_{UG}]{} & FUG \longrightarrow G. \end{array}$$

The map on the right exhibits G as the coequalizer of the two straight arrows on the left. Moreover, the map e exhibits this coequalizer as a reflexive coequalizer. In this sense we see that every group has a *functorial resolution* by free groups. The forgetful functor from Set_T to Set admits a left adjoint F_T which factors T as $T = UF_T$. Similarly, we see that every T -algebra admits a functorial resolution by free T -algebras. After forgetting down to Set these coequalizer diagrams become split coequalizer diagrams [Bor94b, Lemma 4.3.3], i.e., diagrams of the form (2.1). Split coequalizer diagrams have the useful property that they are preserved by *all* functors [Bor94c, Prop. 2.10.2].

Using these functorial resolutions and that a morphism of groups is an isomorphism if and only if it induces an isomorphism between the underlying sets, we can see that the lifted functor $U : \text{Group} \rightarrow \text{Set}_T$ is essentially surjective. By applying the functorial resolution again and (2.2) one can now see that this functor is full and faithful and $\text{Group} \simeq \text{Set}_T$.

These arguments are completely general:

Theorem 2.4 (Barr-Beck/Monadicity). Any functor $U : \mathcal{D} \rightarrow \mathcal{C}$, which admits a left adjoint F , lifts to a functor to the category of $T = UF$ -algebras in \mathcal{C} . Moreover this functor is an equivalence of categories if and only if

- (a) U reflects isomorphisms, i.e., a map f in \mathcal{D} is an isomorphism if and only if Uf is.
- (b) If U takes a pair of arrows of the form (2.3), where G is assumed to be a T -algebra, to a split coequalizer, then the pair of arrows in (2.3) admits a coequalizer which is preserved by U .

Proof. This version of the Barr-Beck theorem is a slight variation of [Bor94b, 4.4.4]. Here we assume the existence of a left adjoint, which does not appear there, and our condition (b) is slightly weaker than what is assumed there. But Borceux's argument applies without change. \square

Theorem 2.4 can be used to identify many categories as categories of algebras over a monad. Since we want \mathcal{C}_T to have an ample supply of colimits and limits for constructing model structures we postpone introducing these examples for a moment so that we can record when such constructions exist.

Proposition 2.5. [Bor94b, 4.3.1, 4.3.2] Suppose T is a monad acting on \mathcal{C} , then

- (1) The forgetful functor $U: \mathcal{C}_T \rightarrow \mathcal{C}$ creates all limits.
- (2) The forgetful functor $U: \mathcal{C}_T \rightarrow \mathcal{C}$ creates all colimits which commute with T in \mathcal{C} .

Proposition 2.6. [EKMM97, Prop. II.7.4] Suppose \mathcal{C} is cocomplete and T commutes with reflexive coequalizers, then \mathcal{C}_T is cocomplete and the forgetful functor creates all reflexive coequalizers.

Alternatively, if we suppose that \mathcal{C} is bicomplete and T preserves κ -filtered colimits for some regular cardinal κ then \mathcal{C}_T is bicomplete by [Bor94b, 4.3.6]. We often want T , or equivalently U , to preserve *both* filtered colimits and reflexive coequalizers (for some examples where this does not hold see [Bor94b, §4.6]). In such a case we can apply the following useful form of the Barr-Beck theorem provided we restrict to *locally presentable categories* [Bor94b, §5.2].

Proposition 2.7. Suppose $U: \mathcal{D} \rightarrow \mathcal{C}$ is a functor between two locally presentable categories such that

- (a) U preserves limits,
- (b) U reflects κ -filtered colimits for some regular cardinal κ ,
- (c) and U creates reflexive coequalizers,

then U admits a left adjoint F , \mathcal{D} is equivalent to the category of $T = UF$ -algebras in \mathcal{C} , and T commutes with reflexive coequalizers and κ -filtered colimits.

The above results illustrate the importance of reflexive coequalizers and filtered colimits in \mathcal{C}_T . These are particular examples of *sifted* colimits, which are colimits indexed over \mathcal{I} such that the diagonal map $\mathcal{I} \rightarrow \mathcal{I} \times \mathcal{I}$ is final. Sifted colimits can also be characterized as those colimits which commute with finite products in \mathcal{Set} . One of the main results of [ARV10, 2.1] is that if \mathcal{C} is finitely cocomplete then T commutes with all (κ) -sifted colimits if and only if T commutes with all reflexive coequalizers and (κ) -filtered colimits.

2.2. Algebraic theories. Monads which commute with sifted colimits arise naturally in the study of algebraic theories in the sense of Lawvere [Law63]. Recall that an (algebraic) theory is a category \mathcal{T} equipped with a product preserving functor $i: \mathcal{FinSet}^{\text{op}} \rightarrow \mathcal{T}$ which is essentially surjective. If we let $\underline{n} \in \mathcal{FinSet}^{\text{op}}$ be a set with n elements, then, since we are working in the opposite category, $\underline{n} \cong \underline{1}^{\times n}$. So \mathcal{T} is equivalent to a category whose objects are $i(\underline{1})^n_{n \in \mathbb{N}}$. If \mathcal{C} is a category with finite products, a \mathcal{T} -model in \mathcal{C} is a product preserving functor $A: \mathcal{T} \rightarrow \mathcal{C}$. The collection of \mathcal{T} -models in \mathcal{C} forms a category $\mathcal{T}\text{-}\mathcal{C}$ where the morphisms are natural transformations.

We should think of \mathcal{T} as encoding the operations on an object of $\mathcal{T}\text{-}\mathcal{C}$. For example, suppose k is a commutative ring and define a theory \mathcal{T} as the subcategory of the opposite category of k -algebras whose n th object $i(\underline{n})$ is the free k -algebra $k\langle x_1, \dots, x_n \rangle$.

Note that for each k -algebra A , we obtain a \mathcal{T} -model in \mathcal{Set} by

$$i(\underline{n}) \mapsto k\text{-}\mathcal{Alg}(k\langle x_1, \dots, x_n \rangle, A) \cong A^n.$$

Conversely, if $A \in \mathcal{T}\text{-}\mathcal{Set}$ we can identify A with the set $A(i(\underline{1}))$ equipped with the operations encoded by the functor A . For example, consider the maps in

$$\mathcal{T}(i(\underline{2}), i(\underline{1})) \cong k\text{-}\mathcal{Alg}(k\langle x_1 \rangle, k\langle x_1, x_2 \rangle)$$

which send x_1 to $x_1 + x_2$ and $x_1 \cdot x_2$ respectively. These two maps define respective natural operations

$$(-) + (-): A(i(\underline{1}))^2 \rightarrow A(i(\underline{1}))$$

$$(-) \cdot (-): A(i(\underline{1}))^2 \rightarrow A(i(\underline{1})).$$

The first map is commutative since $x_1 + x_2 = x_2 + x_1$, while the latter generally is not. By combining maps in \mathcal{T} we can see that the latter operation will distribute over the former. All of these operations and their relations coming from \mathcal{T} show that $A(i(\underline{1}))$ is a k -algebra.

Example 2.8.

- (1) Let \mathcal{T}_{Gp} be the category whose objects are indexed by natural numbers and whose morphisms are

$$\mathcal{T}_{\text{Gp}}(m, n) = \text{Group}(F\{n\}, F\{m\}),$$

where $F\{m\}$ is the free group on m elements. If we let i be the functor which takes a finite set X to the element of \mathcal{T} labeled by $|X|$ then $\mathcal{T}_{\text{Gp}}\text{-Set}$ is equivalent to the category of groups.

- (2) Let G be a group and let \mathcal{T}_G be the theory defined as in (1) but with

$$\mathcal{T}_G(m, n) = G\text{-Set}(F\{n\}, F\{m\}),$$

where $F\{m\}$ is the free G -set on m elements, then $\mathcal{T}_G\text{-Set}$ is equivalent to the category of G -sets.

- (3) Let k be a commutative ring and $\mathcal{T}_{\text{Ass}_k}$ be the theory defined as in (1) but with

$$\mathcal{T}_{\text{Ass}_k}(m, n) = \text{Ass}\mathcal{A}[g_k](F\{n\}, F\{m\}),$$

where $F\{m\}$ is the free associative k -algebra on m elements, then $\mathcal{T}_{\text{Ass}_k}\text{-Set}$ is equivalent to the category of associative k -algebras.

- (4) Let k be a commutative ring and $\mathcal{T}_{\text{Lie}_k}$ be the theory defined as in (1) but with

$$\mathcal{T}_{\text{Lie}_k}(m, n) = \text{Lie}_k(F\{n\}, F\{m\}),$$

where $F\{m\}$ is the free Lie algebra over k on m elements, then $\mathcal{T}_{\text{Lie}_k}\text{-Set}$ is equivalent to the category of Lie algebras over k .

The list in Example 2.8 is far from comprehensive and is limited only by the authors' imagination and the readers' patience.

If \mathcal{T} is a theory, we obtain \mathcal{T} -models $\mathcal{T}\{\underline{m}\}$ in Set by setting $\mathcal{T}\{\underline{m}\}(-) = \mathcal{T}(\underline{m}, -)$, which we can think of as the free objects on a set of m -elements. This construction lifts to a (covariant!) functor $\mathcal{T}\{-\}: \text{FinSet} \rightarrow \mathcal{T}\text{-Set}$. Since Set is the closure of FinSet under sifted colimits, and sifted colimits commute with products in Set we see that we can canonically prolong this to a functor from Set . This functor admits a forgetful right adjoint given by evaluating at $i(\underline{1})$.

Just as in Section 2.1, we can compose these adjoints to obtain a monad $T = U\mathcal{T}\{-\}$ on Set . More explicitly, the formula for the left Kan extension shows

$$(2.9) \quad TX = \int^{n \in \text{FinSet}} X^n \times \mathcal{T}(i(\underline{n}), i(\underline{1})).$$

Since $\mathcal{T}\text{-Set}$ is locally presentable and $U\mathcal{T}\{-\}$ preserves sifted colimits we can apply Proposition 2.7 and see that $\mathcal{T}\text{-Set}$ is equivalent to the category of $T = U\mathcal{T}\{-\}$ algebras in Set .

Given a T -algebra $X \in \mathcal{T}\text{-Set}$ we can consider the category $(\mathcal{T}\text{-Set})_X$ of algebras over X . Generally this can not be realized as the category of Set -valued models for an algebraic theory. However it can be realized as the category of models of a *graded* theory $\mathcal{T} \downarrow X$. This additional generality will prove useful for identifying the E_2 term of the T -algebra spectral sequence with André-Quillen cohomology in Theorem 5.6.

Definition 2.10. For a set S of gradings, an S -graded theory \mathcal{T} is a category \mathcal{T} equipped with a product preserving functor $i: (\text{FinSet}^S)^{\text{op}} \rightarrow \mathcal{T}$ which is essentially surjective. If $(x_s)_{s \in S}$ is an object of FinSet^S , and each x_s is a set with t_s elements, then $x_s \cong [s]^{\times t_s}$ in $\text{FinSet}^{\text{op}}$, where $[s]$

denotes the 1-element set indexed by s . Therefore the object set of \mathcal{T} is, up to equivalence, in bijection with a set $i(t)_{t \in \mathbb{N}[S]}$ where each t has the form $\sum_{s \in S} t_s[s]$ for some $t_s \in \mathbb{N}$ and

$$i(t) \cong \prod_{s \in S} (i([s]))^{t_s}.$$

Here are two prototypical examples.

Example 2.11.

- (1) The category of \mathbb{Z} -graded abelian groups is the category of $\mathcal{T}_{\text{Ab}}^{\mathbb{Z}}$ -models in Set where $\mathcal{T}_{\text{Ab}}^{\mathbb{Z}}$ is the \mathbb{Z} -graded theory defined as follows: For $j \in \mathbb{Z}$ let $\mathbb{Z}^{\oplus t}[j]$ denote the free abelian group on t elements concentrated in degree j . If $i(t)$ and $i(t')$ are two elements of $\mathcal{T}_{\text{Ab}}^{\mathbb{Z}}$ with

$$\begin{aligned} t &= \sum_{j \in S} t_j[j] \\ t' &= \sum_{k \in S} t'_k[k] \end{aligned}$$

then we define

$$\mathcal{T}_{\text{Ab}}^{\mathbb{Z}}(i(t), i(t')) := \mathcal{A}bGroup^{\mathbb{Z}} \left(\bigoplus_{k \in \mathbb{Z}} \mathbb{Z}^{t'_k}[k], \bigoplus_{j \in \mathbb{Z}} \mathbb{Z}^{t_j}[j] \right) \cong \prod_{k \in \mathbb{Z}} \mathcal{A}bGroup \left(\mathbb{Z}^{t'_k}[k], \mathbb{Z}^{t_k}[k] \right).$$

- (2) Fixing a \mathbb{Z} -graded abelian group A , the category of \mathbb{Z} -graded abelian groups over A is the category of Set -valued models for a theory $\mathcal{T}_{\text{Ab}}^{\mathbb{Z}} \downarrow A$ graded by $S = \coprod_{j \in \mathbb{Z}} S_j$ where

$$S_j = \mathcal{A}bGroup^{\mathbb{Z}}(\mathbb{Z}[j], A).$$

The objects of this theory can be identified with the graded abelian groups over A which are finite direct sums of graded abelian groups of the form $\mathbb{Z}^n[j]$.

The theories in Example 2.8 can all be extended to the graded case similarly and in general overcategories coming from graded theories are graded theories.

Remark 2.12. The category of models in Set of an S -graded theory \mathcal{T} admits a forgetful functor to $\text{Set}^S \cong \prod_S \text{Set}$ by evaluation. This functor is finitary (meaning it preserves countable filtered colimits) and monadic (meaning it satisfies Proposition 2.7) with associated monad T , so the category of \mathcal{T} -models in Set is equivalent to the category of T -algebras in Set^S .

This construction is part of a correspondence demonstrated in [ARV11, App. A] between categories of Set -valued models over S -graded theories and finitary monadic categories over Set^S . Their results can in turn be used to show a correspondence between the latter category and algebraic categories in the sense of [Qui69, Qui70].

The following result will help us in Proposition 3.5 connect the machinery of algebraic theories to André-Quillen cohomology.

Proposition 2.13. Let \mathcal{T} be a graded theory and X a \mathcal{T} -model in Set . Let $\mathcal{T} \downarrow X$ be the S -graded theory, whose category of models is $(\mathcal{T}\text{-Set})_{\downarrow X}$, the category of objects over X in $\mathcal{T}\text{-Set}$. Then there is an S -graded theory $(\mathcal{T} \downarrow X)_{\text{ab}}$ such that the category of $(\mathcal{T} \downarrow X)_{\text{ab}}$ -models in Set , $(\mathcal{T} \downarrow X)_{\text{ab}}\text{-Set}$, is equivalent to the category of abelian group objects in $(\mathcal{T} \downarrow X)\text{-Set}$.

These two categories are monadic over Set^S with associated monads $(T \downarrow X)_{\text{ab}}$ and $T \downarrow X$. The forgetful functor

$$\text{Set}_{(T \downarrow X)_{\text{ab}}}^S \rightarrow \text{Set}_{T \downarrow X}^S$$

is monadic with left adjoint Ab .

Proof. We have already noted that the category of models over X is a graded theory. Let S denote the grading for this theory. As a consequence of [Bor94b, Thm. 3.11.3] the category of abelian group objects in the category of models in $\mathcal{S}et$ for a theory is a category of models for a new theory. As noted in [Bla08, 3.3] this argument passes to the S -graded case, mutatis mutandis, to yield an S -graded theory for the abelian group objects.

As these are both locally presentable categories and limits and sifted colimits in these categories are calculated in $\mathcal{S}et^S$ we can apply Proposition 2.7 to complete the proof. \square

2.3. Simplicial categories of T -algebras. The theory of T -algebra limits and colimits from Section 2.2 admits a straightforward extension to the enriched context. For general background on enriched categories and functors between them the reader is encouraged to consult [Bor94a, §6.2] or [Kel05].

Since we are interested in studying the *space* of maps between two T -algebras, we give this extension in the case that \mathcal{C} is a simplicial category. To obtain categorical information analogous to the previous section we will replace all of our categories with simplicial categories, all of our functors with simplicial functors, and all of our natural transformations with simplicial natural transformations. Note that one can regard any topological category as a simplicial category via the symmetric monoidal functor Sing_* .

Recall that we require a simplicial category \mathcal{C} to have a tensor bifunctor

$$\otimes : \mathcal{C} \times s\mathcal{S}et \rightarrow \mathcal{C}.$$

This is related to the simplicial mapping functor $\mathcal{C}(-, -)$ and the simplicial cotensor $(-)^-$ via the following adjunction isomorphisms

$$s\mathcal{S}et(K, \mathcal{C}(C, D)) \cong \mathcal{C}(C \otimes K, D) \cong \mathcal{C}(C, D^K).$$

Proposition 2.14. Suppose that

- (a) \mathcal{C} is a bicomplete simplicial category.
- (b) T is a simplicial monad acting on \mathcal{C} .
- (c) T commutes with either
 - (i) reflexive coequalizers or
 - (ii) filtered colimits.

Then \mathcal{C}_T is a bicomplete simplicial category such that

- (1) The forgetful functor $\mathcal{C}_T \rightarrow \mathcal{C}$ creates limits and cotensors.
- (2) The simplicial tensor is constructed as follows:

$$(2.15) \quad X \otimes_T V = \text{coeq} \left[F_T(TUX \otimes V) \xrightarrow[\mu \circ \alpha]{F_T(\mu \otimes V)} F_T(UX \otimes V) \right].$$

$\begin{array}{c} \xrightarrow{F_T(e \otimes V)} \\ \xleftarrow{F_T(\mu \otimes V)} \end{array}$

Here $\alpha : F_T(TUX \otimes V) \rightarrow F_T(UX \otimes V)$ is adjoint to the assembly map $TUX \otimes V \rightarrow T(UX \otimes V)$.

Proof. First we check that \mathcal{C}_T is bicomplete: By Proposition 2.5 \mathcal{C}_T is complete. Under hypothesis (c.i) we can apply Proposition 2.6 to see that \mathcal{C}_T is cocomplete. When hypothesis (c.ii) holds, cocompleteness follows from [Bor94b, Prop. 4.3.6].

The hom spaces of \mathcal{C}_T are defined by taking the equalizer, in $s\mathcal{S}et$, of the diagram in (2.2). The fact that U creates cotensors appears in [EKMM97, §VII Prop. 2.10] In order for the adjunctions to hold the tensor must be defined by (2.15) and we see that cotensors are created via U . \square

Note that under the hypotheses of Proposition 2.14, if T commutes with reflexive coequalizers we can compute the simplicial tensor in \mathcal{C} .

Graded algebraic theories are extended similarly to the simplicial context: Regarding the category of finite sets as a simplicially enriched category with discrete mapping objects, a simplicial algebraic theory is just a product preserving functor $(\mathcal{F}inSet^S)^{op} \rightarrow \mathcal{T}$ to a simplicially enriched category \mathcal{T} which is essentially surjective as an ordinary functor. Similarly, a T -model in a simplicially enriched category \mathcal{C} with finite products is just a product preserving simplicial functor $\mathcal{T} \rightarrow \mathcal{C}$.

Example 2.16. Each of the examples listed in Example 2.8 and their graded counterparts naturally defines a simplicial theory. The \mathcal{T} -models in simplicial sets are respectively equivalent to the categories of simplicial groups, simplicial abelian groups, et cetera.

Proposition 2.17. Let T be the simplicial monad acting on $sSet^S$ associated to an S -graded simplicial algebraic theory \mathcal{T} . Then the category $sSet_T^S$ of T -algebras is a bicomplete simplicial category with tensor defined by (2.15).

If $S = *$ and \mathcal{T} is an ordinary theory regarded as a constant simplicial theory then for $K \in sSet$ and $X \in sSet_T$ we have the identification

$$(X \otimes K)_n = \coprod_{k \in K_n} X_n.$$

2.4. Monads from operads. A *symmetric sequence* in $sSet$ is a sequence

$$C = \{C(n)\}_{n \geq 0}$$

of spaces such that $C[n]$ has a right action by Σ_n . A map of symmetric sequences is a levelwise equivariant map.

For the remainder of this section we assume that \mathcal{C} is simplicial symmetric monoidal category with tensor \otimes such that:

- (a) \otimes distributes over countable coproducts in \mathcal{C} and
- (b) there is a symmetric monoidal functor $i : sSet \rightarrow \mathcal{C}$, such that the tensor of a space K and an object X of \mathcal{C} is defined by $iK \otimes X$.

Now given a symmetric sequence C , we have an associated functor $T_C : \mathcal{C} \rightarrow \mathcal{C}$ defined on objects by

$$T_C(X) = \coprod_{n \geq 0} C(n) \otimes_{\Sigma_n} X^{\otimes n}.$$

A map of symmetric sequences yields a natural transformation of functors, and this construction yields a functor from symmetric sequences to endofunctors of \mathcal{C} .

There is an external product on symmetric sequences, which we again denote by \otimes :

$$(C \otimes D)(n) = \coprod_{i+j=n} C(i) \times D(j) \times_{\Sigma_i \times \Sigma_j} \Sigma_n$$

Since the symmetric monoidal structure on \mathcal{C} distributes over coproducts we see:

$$T_C \otimes T_D \cong T_{C \otimes D}.$$

Now we define the circle product by:

$$(C \circ D)(n) = C(n) \times_{\Sigma_n} D^{\otimes n}.$$

We can now check that the construction $C \mapsto F_C$ defines a monoidal functor from symmetric sequences under the circle product to endofunctors under composition. An operad \mathcal{O} is a symmetric sequence which is a monoid for the circle product; the associated endofunctor is then a monad (see [Rez97, §11] for additional details).

Remark 2.18. The category of operads in $sSet$ can be constructed as the category of $sSet$ -valued models for a graded simplicial algebraic theory. As in [Rez96, App. A] one can construct the free monoid with respect to the circle product on a symmetric sequence. Regarding the Σ_n set $\Sigma_n \times \underline{i}$ as a symmetric sequence concentrated in degree n (and the empty set elsewhere), we can apply this free construction to the symmetric sequences $\{\Sigma_n \times \underline{i}\}_{(n,i) \in \mathbb{N} \times \mathbb{N}}$ to define an \mathbb{N} -graded algebraic theory, whose algebras are operads in Set . The category of simplicial operads is the associated category of models in $sSet$.

The following standard result gives criteria for identifying when the category of algebras over an operad is simplicially enriched.

Proposition 2.19. Suppose that \mathcal{C} is a bicomplete simplicial symmetric monoidal category \mathcal{C} such that:

- (a) There is a symmetric monoidal functor $i: sSet \rightarrow \mathcal{C}$ defining the simplicial tensor.
- (b) The monoidal product in \mathcal{C} commutes with countable coproducts and either
 - (i) reflexive coequalizers or
 - (ii) filtered colimits.

Then for any operad \mathcal{O} of simplicial sets, the category of \mathcal{O} -algebras in \mathcal{C} is a bicomplete simplicial category.

The hypotheses concerning colimits for this proposition hold whenever the symmetric monoidal structure comes from a *closed* symmetric monoidal structure and hence distributes over all colimits. For example, simplicial sets, simplicial abelian groups, and simplicial R -modules all satisfy the conditions of Proposition 2.19 with their cartesian symmetric monoidal structure. The categories of pointed compactly generated spaces or pointed simplicial sets, each equipped with the smash product, satisfy these conditions. Any of the closed symmetric monoidal categories of spectra satisfy the hypotheses.

3. MODEL STRUCTURES

In Section 3.1 we recall conditions that guarantee that the category of T -algebras has a suitable homotopy theory. After establishing the existence of a model structure, we construct functorial simplicial resolutions of algebras in Section 3.2 and these give rise to the spectral sequence.

Here, we choose to work in the context of simplicial model categories. A disadvantage of this approach is that some of our assumptions—most notably the existence of colimits/limits and the standard issues concerning cofibrancy and fibrancy—should not be strictly necessary (see for example [Lur12, §6.2]).

An advantage of this approach is that the theory is well-developed, well-understood, and relatively straightforward to apply to many categories of interest. We have gathered the relevant results from the literature in the interest of having a single reference for determining whether a category of T -algebras admits a simplicial model structure. The background material for this section can be found in [Qui69, Hov99, Hir03] or the appendices of [Lur09].

3.1. Model structure on \mathcal{C}_T . We will now recall conditions which guarantee the simplicial structure on \mathcal{C}_T is part of a simplicial model structure [Qui67]. These model categories satisfy the following two equivalent forms of Quillen’s corner axiom.

SM7: Given any cofibration $f \in sSet(K, L)$ and fibration $g \in \mathcal{M}(A, B)$, the induced morphism

$$A^L \longrightarrow A^K \times_{B^K} B^L$$

is a fibration, which is a weak equivalence if either f or g is.

SM7a: Given cofibrations $f \in \mathcal{S}et(K, L)$ and $g \in \mathcal{M}(A, B)$, the induced morphism

$$K \otimes B \coprod_{K \otimes A} L \otimes A \longrightarrow L \otimes B$$

is a cofibration, which is a weak equivalence if either f or g is.

Definition 3.1. A monad T acting on a category \mathcal{C} is *Quillen* if

- (a) \mathcal{C} is a simplicial model category.
- (b) T is a simplicial monad acting on \mathcal{C} .
- (c) \mathcal{C}_T has a simplicial model structure such that the forgetful functor $U: \mathcal{C}_T \rightarrow \mathcal{C}$ is a simplicial right Quillen functor.
- (d) A map f of T -algebras is a weak equivalence if and only if Uf is a weak equivalence.

A convenient way to show that T is Quillen is to assume we have a simplicial model structure on \mathcal{C} and induce a model structure on \mathcal{C}_T via F_T . We can do this if \mathcal{C} is cofibrantly generated and T satisfies some mild hypotheses. In this case \mathcal{C} has sets of generating cofibrations I and acyclic cofibrations J which are used to detect fibrations and acyclic fibrations. These sets of maps satisfy smallness hypotheses which are used to apply Quillen's small object argument and prove the lifting axioms.

Suppose that \mathcal{C} is a model category and a functor

$$U: \mathcal{D} \rightarrow \mathcal{C}$$

admits a left adjoint. Then we say that U *right induces* a model structure on \mathcal{D} if \mathcal{D} admits a model structure such that a map f is a fibration (resp. weak equivalence) if and only if Uf is a fibration (resp. weak equivalence).

Theorem 3.2. [Sch07, App. A] Suppose that \mathcal{C} is a cofibrantly generated simplicial model category with generating (acyclic) cofibrations I (resp. J) and $T = UF_T$ is a monad on \mathcal{C} satisfying Proposition 2.14.

If the domains of F_TI (resp. F_TJ) are small relative to F_TI -cells (resp. F_TJ -cells) and applying U to any F_TJ -cell complex yields a weak equivalence in \mathcal{C} then U right induces a cofibrantly generated simplicial model category structure on \mathcal{C}_T .

Remark 3.3. In practice, checking the smallness conditions is relatively easy. In fact, it is automatic when the underlying categories are locally presentable. So most of the work required to apply Theorem 3.2 involves checking that applying U to a F_TJ -cell diagram yields a weak equivalence. Assuming these smallness conditions, this can be verified (see [Sch99, Lem. B2]) by showing the following two properties are satisfied.

- (1) There is a 'fibrant replacement' functor $Q: \mathcal{C}_T \rightarrow \mathcal{C}_T$ and a natural transformation $\text{Id} \rightarrow Q$ such that for all $X \in \mathcal{C}_T$, the natural map $UX \rightarrow UQX$ is a fibrant replacement.
- (2) If UX is fibrant then applying U to the canonical factorization $X \rightarrow X^{\Delta^1} \rightarrow X^{\partial\Delta^1} \cong X \times X$ of the diagonal yields a weak equivalence followed by a fibration.

The second property follows from the fact that U preserves cotensors and \mathcal{C} is a simplicial model category. For the first property one can sometimes show that the fibrant replacement functor \mathcal{C} lifts to an endomorphism of \mathcal{C}_T . This is automatic if every object is fibrant in \mathcal{C} . Since the two fibrant replacement functors Ex^∞ and $\text{Sing}_*|-|$ on simplicial sets are product preserving we can use either of them as fibrant replacement functors for simplicial T -algebras associated to a (graded) theory to obtain the following propositions.

Proposition 3.4 (Comp. [Sch01, Thm. 3.1]). Each of the monads coming from a (graded) algebraic theory on simplicial sets (such as those in Example 2.16 or Remark 2.18) is Quillen.

Proposition 3.5. Let \mathcal{T} be a graded theory and X a model in $sSet$ for this theory. Then there is an S -graded theory whose models in $sSet$, which we will denote by A , is equivalent to the category of abelian group objects in B , the the category of $(\mathcal{T}$ -models in $sSet$) over X (which is also an S -graded theory).

These two categories are monadic over $sSet^S$ with associated monads T_A and T_B and the inclusion

$$\iota: sSet_{T_A}^S \rightarrow sSet_{T_B}^S$$

is monadic with left adjoint Ab and its associated monad is Quillen.

Proof. The argument from Proposition 2.13 can be applied here leaving us to check that ι is a right Quillen functor. In both categories a map is a weak equivalence (resp. fibration) if and only if it is a weak equivalence (resp. fibration) in $sSet^S$. Since the forgetful functor $sSet_{T_A}^S$ to $sSet^S$ factors through ι the result follows. \square

Definition 3.6. Given a simplicial model category \mathcal{C} we define the derived mapping space

$$\mathcal{C}^d(X, Y) = \mathcal{C}(X', Y')$$

where X' is a cofibrant replacement of X and Y' is a fibrant replacement of Y . We now define

$$ho\mathcal{C}(X, Y) = \pi_0 \mathcal{C}^d(X, Y).$$

It follows easily from Axiom [SM7] that the derived mapping space is well-defined up to weak equivalence and therefore $ho\mathcal{C}(X, Y)$ is well-defined. That this definition agrees with other constructions of the set of homotopy classes of morphisms, and that these hom-sets assemble into a homotopy category $ho\mathcal{C}$ can be found in [GJ99, Prop. 3.10] for example.

Proposition 3.7. Suppose that the forgetful functor $U: \mathcal{C}_T \rightarrow \mathcal{C}$ is a Quillen right adjoint. Then the monad T induces a monad hT on $ho\mathcal{C}$ such that the forgetful functor $ho(\mathcal{C}_T) \rightarrow ho\mathcal{C}$ factors through $(ho\mathcal{C})_{hT}$.

Proof. Quillen adjoints induce adjoints between the homotopy categories and consequently a monad action on $ho\mathcal{C}$ given by the composite. The right adjoint between the homotopy categories always lands in the category of algebras over this monad. \square

Definition 3.8. Let \mathcal{T} be an S -graded theory and X a \mathcal{T} -model in $sSet$. Let

$$\iota: sSet_{(T \downarrow X)_{\text{ab}}}^S \rightarrow sSet_{T \downarrow X}^S$$

be the Quillen right adjoint from Proposition 3.5.

If M is in $Set_{(T \downarrow X)_{\text{ab}}}^S$ which we regard as a constant simplicial object and $Y \in sSet_{T \downarrow X}^S$, then the sth André-Quillen cohomology of Y with coefficients in M is defined to be the group

$$H_{AQ, \downarrow X}^s(Y; M) := ho sSet_{T \downarrow X}^S(Y, \iota \Sigma^s M)$$

where $\Sigma^s M$ is the sth suspension of M (see [Qui69]).

3.2. Simplicial resolutions. To construct a spectral sequence computing the homotopy groups of the space $\mathcal{C}_T(X, Y)$ we would like to resolve X , meaning that we replace X by a simplicial T -algebra X_\bullet such that $\mathcal{C}_T(|X_\bullet|, Y) \simeq \mathcal{C}_T^d(X, Y)$.

If T is a monad acting on \mathcal{C} , then applying T levelwise to simplicial objects in \mathcal{C} yields a monad on $s\mathcal{C}$ which we also denote by T .

Definition 3.9. Suppose X is a T -algebra in \mathcal{C} . The *bar resolution* (also called the cotriple resolution) of X is the simplicial T -algebra

$$B_\bullet X = B_\bullet(F_T, T, UX) = B_\bullet(F_T U, F_T U, X)$$

with $B_n X = (F_T U)^{n+1} X$ and face and degeneracy maps induced from the monad structure on $T = UF_T$ and the T -algebra structure on X .

Note that the counit $F_T U X \rightarrow X$ extends to a map of simplicial T -algebras

$$(3.10) \quad \varepsilon: B_\bullet X \rightarrow X$$

where we regard the target as a constant simplicial object. By applying U to (3.10) we obtain a map in $s\mathcal{C}$

$$\varepsilon: T^{\bullet+1} U X \rightarrow U X.$$

We also have a simplicial map

$$e: U X \rightarrow T^{\bullet+1} U X$$

by iterating the unit map $U X \rightarrow T U X$.

For a simplicial T -algebra X , there are two relevant geometric realizations. One is realization in the category of T -algebras, and another is realization in the underlying category. We would like to have conditions under which these two notions coincide, i.e., under which U commutes with geometric realizations.

Remark 3.11. One such condition appears in [EKMM97, p. 197]: If T is given by a coend formula, then U preserves geometric realizations. More precisely, if T is given by a formula such as the one in (2.9), we will show that T commutes with geometric realization and then apply Proposition 3.12 to see that U commutes with geometric realizations. To show that T commutes with geometric realization we use the fact that geometric realization commutes with finite products and Fubini's theorem for iterated coends as follows:

$$\begin{aligned} T|UX_\bullet|_{\mathcal{C}} &= \int^{j \in \text{FinSet}} |UX_\bullet|_{\mathcal{C}}^{|j|} \otimes \mathcal{T}(j, 1) \\ &\cong \int^{j \in \text{FinSet}} |UX_\bullet|_{\mathcal{C}}^{|j|} \otimes \mathcal{T}(j, 1) \\ &= \int^{j \in \text{FinSet}} \left(\int^{n \in \Delta} X_n^{|j|} \otimes \Delta^n \right) \otimes \mathcal{T}(j, 1) \\ &\cong \int^{\Delta} \left(\int^{j \in \text{FinSet}} X_n^{|j|} \otimes \mathcal{T}(j, 1) \right) \otimes \Delta^n \\ &= |TX_\bullet|_{\mathcal{C}}. \end{aligned}$$

Note that the commutation of products with geometric realization plays a key role in the above result. This is easy to verify in the case where \mathcal{C} is simplicial sets, since the geometric realization of a bisimplicial set is isomorphic to its diagonal. The result is non-trivial but still true in the case of compactly generated weak Hausdorff spaces. From these cases one can deduce that the smash product on simplicial objects in pointed simplicial sets, compactly generated pointed spaces, or categories of spectra built from these categories also commutes with geometric realization. As a consequence similarly defined monads will also commute with geometric realization.

Proposition 3.12. Let X_\bullet be a simplicial object in \mathcal{C}_T . If the conditions of Proposition 2.14 are satisfied, so \mathcal{C}_T is a bicomplete simplicial category, and T commutes with geometric realization, then $|UX_\bullet|_{\mathcal{C}}$ is a T -algebra and $|X_\bullet|_{\mathcal{C}_T} \cong |UX_\bullet|_{\mathcal{C}}$ in \mathcal{C}_T . So U commutes with geometric realization.

Proof. Beginning with the canonical presentation of a simplicial T -algebra

$$\begin{array}{c} \xrightarrow{\quad e \quad} \\ F_T T U X_\bullet \rightrightarrows F_T U X_\bullet \longrightarrow X_\bullet, \end{array}$$

we now take the geometric realization in the category of T -algebras and apply U to obtain

$$\begin{array}{ccccc}
 & & e & & \\
 & \swarrow & & \searrow & \\
 U|F_T TUX_\bullet|_{\mathcal{C}_T} & \xrightarrow{\quad} & U|F_T UX_\bullet|_{\mathcal{C}_T} & \longrightarrow & U|X_\bullet|_{\mathcal{C}_T} \\
 \cong \downarrow & & \downarrow \cong & & \parallel \\
 T|TUX_\bullet|_{\mathcal{C}} & \xrightarrow{\quad} & T|UX_\bullet|_{\mathcal{C}} & \longrightarrow & U|X_\bullet|_{\mathcal{C}_T} \\
 \cong \downarrow & & \downarrow \cong & & \downarrow \cong \\
 |TTUX_\bullet|_{\mathcal{C}} & \xrightarrow{\quad} & |TUX_\bullet|_{\mathcal{C}} & \longrightarrow & |UX_\bullet|_{\mathcal{C}}.
 \end{array}$$

□

One can interpret the following result as saying that the bar resolution is indeed a resolution.

Proposition 3.13. Suppose that T is a Quillen monad acting on \mathcal{C} which commutes with geometric realization. Then

$$\varepsilon: |B_\bullet X|_{\mathcal{C}_T} \rightarrow X$$

is a weak equivalence of T -algebras.

Proof. This follows from Proposition 3.12 and the following well known lemma. □

Lemma 3.14. [May72, 9.8] Let $X \in \mathcal{C}_T$. The maps e and ε on realization

$$UX \xrightarrow{e} |T^{\bullet+1}UX|_{\mathcal{C}} \xrightarrow{\varepsilon} UX$$

exhibit UX as a strong deformation retract of $|T^{\bullet+1}UX|_{\mathcal{C}}$ in \mathcal{C} .

3.3. Reedy model structure. To construct our spectral sequence using the bar resolution of X we require that this resolution is homotopically well behaved, that is we will require it to be a Reedy cofibrant simplicial diagram. Note that if we simply take a cofibrant replacement of the bar resolution we will no longer be able to apply Proposition 3.13 to deduce that the geometric realization of our resolution has the correct homotopy type. To show that the bar resolution is Reedy we will use a new and useful trick (Proposition 3.17) which makes use of a closely related *almost simplicial* diagram.

Let Δ_0 be the subcategory of Δ with the same objects but whose morphisms are those morphisms of linearly ordered sets which preserve the minimal element. The restriction morphism

$$i^*: \mathcal{C}^{\Delta^{\text{op}}} \rightarrow \mathcal{C}^{\Delta_0^{\text{op}}}$$

takes a simplicial object and forgets the d_0 face maps (those induced by injections missing the minimal element) while retaining all of the other structure. So one can think of a Δ^{op} shaped diagram as an almost simplicial diagram, simply lacking the d_0 face maps.

Definition 3.15. Let X_\bullet be in $\mathcal{C}^{\Delta^{\text{op}}}$ (resp. $\mathcal{C}^{\Delta_0^{\text{op}}}$). The n th *latching object* of X_\bullet is

$$L_n(X_\bullet) = \text{colim}_{[n] \twoheadrightarrow [k]} X_k,$$

where the colimit is indexed over the non-identity surjections in Δ (this is equal to the set of non-identity surjections in Δ_0).

Definition 3.16. Suppose that \mathcal{C} is a model category then the *Reedy* model structure on $\mathcal{C}^{\Delta^{\text{op}}}$ (resp. $\mathcal{C}^{\Delta_0^{\text{op}}}$) is determined by

- (a) $f: X_\bullet \rightarrow Y_\bullet$ is a (Reedy) weak equivalence if $f_n: X_n \rightarrow Y_n$ is a weak equivalence in \mathcal{C} for all $n \geq 0$.

(b) $f: X_\bullet \rightarrow Y_\bullet$ is a (Reedy) cofibration if the induced map

$$X_n \coprod_{L_n X_\bullet} L_n Y_\bullet \rightarrow Y_n$$

is a cofibration in \mathcal{C} for all $n \geq 0$.

To show the bar resolution is Reedy cofibrant in particular cases we will use the following trick:

Proposition 3.17. Suppose \mathcal{C} and \mathcal{D} are model category and $L: \mathcal{D} \rightarrow \mathcal{C}$ is a left Quillen functor. Let X_\bullet be a simplicial diagram in \mathcal{C} and $\iota^* X_\bullet \in \mathcal{C}^{\Delta_0^{\text{op}}}$ its restriction. Suppose that there exists a Reedy cofibrant $\tilde{X}_\bullet \in \mathcal{D}^{\Delta_0^{\text{op}}}$ such that $L\tilde{X}_\bullet \cong \iota^* X_\bullet$. Then X_\bullet is Reedy cofibrant.

Proof. By definition X_\bullet is Reedy cofibrant if for each non-negative n the latching map

$$L_n X_\bullet \rightarrow X_n$$

is a cofibration. Note that the latching object and map depend only on the restriction of X_\bullet to the subcategory $\Delta_{\text{surj}}^{\text{op}}$, where Δ_{surj} consists of all objects $[n]$ but only surjective maps. In particular it suffices to show $\iota^* X_\bullet$ is Reedy cofibrant. Since L is a Quillen left adjoint it commutes with colimits and preserves cofibrations so it takes the Reedy cofibrant \tilde{X}_\bullet to the Reedy cofibrant diagram $L\tilde{X}_\bullet$. Since being cofibrant is invariant under isomorphism the result follows. \square

For a T -algebra $X \in \mathcal{C}_T$, we have a Δ_0^{op} -shaped diagram

$$T^\bullet U X: \Delta_0^{\text{op}} \rightarrow \mathcal{C}$$

where

$$(T^\bullet U X)[n] = T^n U X$$

and the maps $(T^\bullet U X)(s_i)$ and $(T^\bullet U X)(d_i)$ are defined as in the bar construction.

Definition 3.18. Suppose that T is a Quillen monad acting on \mathcal{C} . A T -algebra X is *resolvable* if it is weakly equivalent as T -algebras to a T -algebra \tilde{X} (called its replacement) such that $T^\bullet U \tilde{X}$ is Reedy cofibrant in $\mathcal{C}^{\Delta_0^{\text{op}}}$.

Proposition 3.19. Let T be a Quillen monad acting on \mathcal{C} and X be a resolvable T -algebra with replacement \tilde{X} . Then the bar resolution $B_\bullet \tilde{X}$ is a Reedy cofibrant simplicial T -algebra.

Proof. By assumption $T^\bullet U \tilde{X}$ is Reedy cofibrant in $\mathcal{C}^{\Delta_0^{\text{op}}}$. We obtain the conclusion by applying the left Quillen functor $F_T: \mathcal{C} \rightarrow \mathcal{C}_T$ levelwise to this diagram and using Proposition 3.17. \square

The remainder of this section is devoted to proving various technical results which will assist in determining when a T -algebra is resolvable.

3.3.1. Monads on diagrams of simplicial sets.

Lemma 3.20. Let S be a set and $\mathcal{C} = s\text{Set}^S$ equipped with the product model structure. Then any diagram $X_\bullet: \Delta_0^{\text{op}} \rightarrow \mathcal{C}$ is Reedy cofibrant.

Proof. We will show that Δ_0 is Eilenberg-Zilber [BR12a, Definition 4.1], i.e., Δ_0 satisfies:

(EZ1) For all surjections $\sigma: [n+m] \rightarrow [n]$ in Δ_0 , the set of sections

$$\Gamma(\sigma) = \{\tau \in \Delta_0 \mid \sigma\tau = \text{id}_{[n]}\}$$

is nonempty.

(EZ2) For any two distinct surjections $\sigma_1, \sigma_2: [n+m] \rightarrow [n]$, the sets of sections $\Gamma(\sigma_1)$ and $\Gamma(\sigma_2)$ are distinct.

By [BR12a, 4.2], every Eilenberg-Zilber Reedy category is *elegant* ([BR12a, Definition 3.5]) and by [BR12a, 3.15] the product and Reedy model structures agree on categories of elegant diagrams in $\mathcal{C} = sSet^S$. In particular, the object X_\bullet will be Reedy cofibrant because the cofibrations in the product model structure are the levelwise cofibrations and every simplicial set is cofibrant.

To verify (EZ1), we note that if σ is a surjection in Δ_0 , it is also a surjection in Δ and therefore has a section $\tau \in \Delta$. Define

$$\tau'(i) = \begin{cases} 0 & \text{if } i = 0 \\ \tau(i) & \text{if } i > 0 \end{cases}$$

The map τ' is certainly order-preserving because 0 is minimal and τ is order-preserving. It is also a section of σ because $\sigma(0) = 0$ and $\sigma(\tau(i)) = i$ for $i > 0$. Moreover, $\tau'(0) = 0$ so $\tau' \in \Delta_0$.

To verify (EZ2), suppose that σ_1 and σ_2 are two distinct surjections of Δ_0 . Then there is a minimal j such that $\sigma_1(j) \neq \sigma_2(j)$. Note that j must be positive. Without loss of generality, we may assume $\sigma_1(j) < \sigma_2(j)$. Define τ_2 by letting $\tau_2(i)$ be the minimal element of $\sigma_2^{-1}(i)$ for each i ; this defines an order preserving section of σ_2 with $\tau_2(0) = 0$, so $\tau_2 \in \Delta_0$. By minimality of j , we must have $\sigma_1(i) = \sigma_2(i)$ for $i < j$, and hence $\tau_2(\sigma_2(j)) = j$. Therefore

$$(\sigma_1 \tau_2)(\sigma_2(j)) = \sigma_1(j) \neq \sigma_2(j),$$

so τ_2 is not a section of σ_1 . □

Proposition 3.21. Suppose that T is a Quillen monad acting on $sSet^S$ then any T -algebra is resolvable.

Proof. This is an immediate application of Lemma 3.20. □

3.3.2. Cellular monads.

Proposition 3.22. Let \mathcal{C} be a cofibrantly generated model category where relative cell complexes are monomorphisms and let $X_\bullet \in \mathcal{C}^{\Delta_0^{\text{op}}}$ be an objectwise cellular diagram such that each degeneracy s_i is a subcellular inclusion. Then the latching maps of X_\bullet are cellular inclusions so X_\bullet is Reedy cofibrant.

Proof. We modify the proof of [EKMM97, X.2.7] to show inductively that [EKMM97, X.2.5] is a pushout-pullback diagram of subcell complexes defined as unions of the subcell complexes given by the degeneracies. Such unions are well-defined because relative cell complexes are monomorphisms [Hir03, Prop. 10.6.10]. □

Proposition 3.23. Let T be a Quillen monad acting on a cofibrantly generated model category \mathcal{C} . Suppose that relative cell complexes in \mathcal{C} are monomorphisms and suppose that for any cellular object M , TM is cellular and the natural unit map $M \rightarrow TM$ is a cellular inclusion. If X is a T -algebra and is weakly equivalent as a T -algebra to some \tilde{X} such that $U\tilde{X}$ is cellular, then X is resolvable with replacement \tilde{X} .

Proof. This is an immediate application of Proposition 3.22. □

3.3.3. Monads whose unit maps are inclusions of summands.

Proposition 3.24. Let $X_\bullet \in \mathcal{C}^{\Delta_0^{\text{op}}}$ be a diagram in a pointed model category \mathcal{C} such that X_0 is cofibrant and each degeneracy s_i is a cofibration and the inclusion of a summand. Then the latching maps of X_\bullet are cofibrations and summand inclusions, and therefore X_\bullet is Reedy cofibrant.

Proof. First let $X_\infty = \text{colim}_i X_i$ where X_i maps to X_{i+1} via s_0 . Let A_∞ be a set such that

$$X_\infty = \bigvee_{\alpha \in A_\infty} Y_\alpha$$

where Y_α cannot be written as a nontrivial coproduct. Since each X_n is an inclusion of a summand of X_{n+1} , there are subsets $A_n \subset A_\infty$ such that

$$X_n = \bigvee_{\alpha \in A_n} Y_\alpha.$$

The degeneracies $X_{n-i} \rightarrow X_n$ are summand inclusions and therefore are induced by subset inclusions $A_{n-i} \rightarrow A_n$ which we call the set-level degeneracies. We can now identify the degenerate simplices $X_n^{\text{dg}} = \bigvee_{\alpha \in A'_n} Y_\alpha$ where A'_n is the union (i.e., colimit) of all the A_{n-i} under these set-level degeneracies for $1 \leq i \leq n$. This union of sets indexes the colimit of objects yielding the latching object, so we can identify X_n^{dg} with $L_n X_\bullet$ and the latching map with that induced by the inclusion $A'_n \rightarrow A_n$.

Let X_n^{nd} be the complementary summand of $L_n X_\bullet$ —this is the nondegenerate part of X_n . To see that the latching map is a cofibration we begin by observing that it is a coproduct of the identity map on the latching object with the map from the initial object into X_n^{nd} . Now X_n is cofibrant because each of the degeneracies are cofibrations and X_0 is cofibrant. The retract X_n^{nd} is therefore cofibrant, and hence the latching map is a coproduct of cofibrations. \square

Proposition 3.25. Let T be a Quillen monad acting on a pointed category \mathcal{C} . Suppose that for any cofibrant object M the natural unit map $M \rightarrow TM$ is a cofibration and inclusion of a summand. If X is a T -algebra and is weakly equivalent as a T -algebra to some \tilde{X} such that $U\tilde{X}$ is cofibrant, then X is resolvable with replacement \tilde{X} .

Proof. This is an immediate application of Proposition 3.24. \square

4. THE SPECTRAL SEQUENCE AND EXAMPLES

4.1. Proof of Theorem 1.1. Now we recall and prove the central theorem of this paper:

Theorem. Suppose T is a monad acting on a simplicial category \mathcal{C} and X and Y are T -algebras such that:

- (a) T is Quillen,
- (b) T is commutes with geometric realization,
- (c) and X is resolvable with replacement \tilde{X} .

Let $U: \mathcal{C}_T \rightarrow \mathcal{C}$ denote the forgetful functor from the category of T -algebras to \mathcal{C} . Then T induces a monad hT on $ho\mathcal{C}$ and there exists an obstruction-theoretic spectral sequence satisfying:

- (1) $E_1^{0,0} = ho\mathcal{C}(UX, UY)$.
- (2) $E_2^{0,0} = (ho\mathcal{C})_{hT}(UX, UY)$. That is, a homotopy class $[f]: UX \rightarrow UY$ survives to the E_2 page if and only if it is a map of hT -algebras in the homotopy category.
- (3) Provided a T -algebra map $\varepsilon: X \rightarrow Y$ to serve as a base point, the spectral sequence conditionally converges to the homotopy of the derived mapping space

$$\pi^s \pi_t(\mathcal{C}^d(T^\bullet U\tilde{X}, UY), \varepsilon) \Rightarrow \pi_*(\mathcal{C}_T^d(X, Y), \varepsilon).$$

- (4) In this case the differentials $d_r[f]$ provide obstructions to lifting $[f]$ to a map of T -algebras.
- (5) The edge homomorphisms

$$\begin{aligned} \pi_0 \mathcal{C}_T^d(X, Y) &\twoheadrightarrow E_\infty^{0,0} \\ &\hookrightarrow E_2^{0,0} = (ho\mathcal{C})_{hT}(UX, UY) \\ &\hookrightarrow E_1^{0,0} = ho\mathcal{C}(UX, UY) \end{aligned}$$

are the corresponding forgetful functors.

- (6) The spectral sequence is contravariantly functorial in $X \in ho(\mathcal{C}_T)$ and covariantly functorial in $Y \in ho(\mathcal{C}_T)$ and T satisfying the hypotheses.

Proof. First, in order for the theorem to make sense there must be a derived mapping space of T -algebras. This follows from the assumption that T is Quillen.

The conclusions of the theorem depend only on the weak equivalence classes of X and Y , so without loss of generality we assume Y is a fibrant T -algebra and, replacing X with \tilde{X} if necessary, that X is a T -algebra such that T^*UX is a Reedy cofibrant diagram in $\mathcal{C}^{\Delta_0^{op}}$. By Proposition 3.19 the bar resolution $B_\bullet X$ is a Reedy cofibrant simplicial T -algebra. Since Y is fibrant and \mathcal{C} is a simplicial model category, applying the mapping space functor $\mathcal{C}_T(-, Y)$ to a Reedy cofibrant simplicial T -algebra yields a Reedy fibrant cosimplicial space. In particular, $\mathcal{C}_T(B_\bullet X, Y)$ is Reedy fibrant.

Applying [Bou89], the totalization tower for this Reedy fibrant cosimplicial space arising from the skeletal filtration on $|B_\bullet X|$ yields an obstruction-theoretic spectral sequence computing the homotopy of the totalization

$$\mathrm{Tot}(\mathcal{C}_T(B_\bullet X, Y)) \cong \mathcal{C}_T(|B_\bullet X|, Y).$$

This spectral sequence conditionally converges provided there exists a base point at which to take homotopy groups. (A list of obstructions to determining such a base point is also provided by the construction; see Remark 4.4.)

Now since $B_\bullet X$ is Reedy cofibrant and \mathcal{C}_T is a simplicial model category, $|B_\bullet X|$ is a cofibrant T -algebra. Since T commutes with geometric realization, Proposition 3.13 shows that the augmentation map

$$|B_\bullet X| \rightarrow X$$

is a weak equivalence of T -algebras. It follows that $\mathcal{C}_T(|B_\bullet X|, Y)$ is a model for $\mathcal{C}_T^d(X, Y)$ and this gives the target of the spectral sequence in (3). Conclusion (4) follows immediately from the conditional convergence of the spectral sequence.

The $E_1^{0,0}$ term of Bousfield's spectral sequence is the set

$$\pi_0 \mathcal{C}_T(B_0 X, Y) = \pi_0 \mathcal{C}_T(F_T U X, Y) \cong \pi_0 \mathcal{C}(U X, U Y).$$

To prove (1) we will show the right-hand side can be identified with morphisms in the homotopy category. This follows if $U X$ is cofibrant and $U Y$ is fibrant since \mathcal{C} is a simplicial model category. These conditions follow from the hypotheses that X is resoluble and that T is Quillen: Because T^*UX is Reedy cofibrant, the zeroth latching map shows that $U X$ is cofibrant. Since T is Quillen, U is a right Quillen functor and therefore $U Y$ is fibrant because Y is fibrant.

The edge homomorphism

$$\pi_0 \mathcal{C}_T^d(X, Y) \rightarrow E_1^{0,0}$$

is induced by restricting along the inclusion

$$\mathrm{sk}_0 |B_\bullet X| = F_T U X \rightarrow |B_\bullet X|$$

which by adjunction gives the second half of (5). The first half follows from the identification of the $E_2^{0,0}$ term in (2).

To prove (2) recall that the $E_2^{0,0}$ term of Bousfield's spectral sequence is defined to be the equalizer of the two face maps

$$\pi_0 \mathcal{C}_T(B_0 X, Y) \rightrightarrows \pi_0 \mathcal{C}_T(B_1 X, Y).$$

We again use the adjunction and the fact that T^*UX is Reedy cofibrant to see that the diagram above is isomorphic to

$$ho\mathcal{C}(U X, U Y) \rightrightarrows ho\mathcal{C}(T U X, U Y),$$

whose equalizer is, by definition, $(ho\mathcal{C})_{hT}(U X, U Y)$ (see (2.2) and Proposition 3.7). In other words, a map lifts to $E_2^{0,0}$ precisely if it is a homotopy T -algebra map.

Provided a base point ϵ for the spectral sequence, or even a point that lifts to Tot^2 (see Remark 4.4), the E_1 page of this spectral sequence is given by applying π_t to the spaces $\mathcal{C}_T(B_s X, Y)$ and normalizing as in [Bou89, §2]. The E_2 term can be identified with the cohomotopy of this graded cosimplicial object which is typically denoted as follows:

$$E_2^{s,t} = \pi^s \pi_t(\mathcal{C}_T(B_\bullet X, Y), \epsilon).$$

By adjunction we have

$$\mathcal{C}_T(B_n X, Y) = \mathcal{C}(F_T T^n U X, Y) \cong \mathcal{C}(T^n U X, U Y).$$

As in the previous steps, the right-hand side is a model for the derived mapping space since $U Y$ is fibrant and $T^\bullet U X$ is Reedy cofibrant. This completes the proof of (3).

To see that the spectral sequence is functorial with respect to maps in Y , where Y is assumed to be fibrant, and X such that $T^\bullet U X$ is Reedy cofibrant. The former is obvious and the latter follows from the naturality of the bar construction.

To check functoriality in T we suppose that we have the following diagram of adjunctions

$$\begin{array}{ccc} \mathcal{C} & \xrightleftharpoons[F_{T_2}]{U_2} & \mathcal{C}_{T_2} \\ & \searrow F_{T_1} \quad \nearrow F_{T_3} & \\ & \mathcal{C}_{T_1} & \end{array}$$

U_1 (left arrow), U_3 (right arrow)

where $F_{T_2} = F_{T_3} F_{T_1}$ and $U_{T_2} = U_1 U_3$. We assume that all of the adjunctions are simplicial Quillen adjunctions and their associated monads satisfy the hypotheses of the theorem. Moreover we suppose that X has Reedy cofibrant resolutions with respect to T_1 and T_2 simultaneously. To obtain a map between the spectral sequences corresponding to the map of mapping spaces:

$$\mathcal{C}_{T_2}(X, Y) \xrightarrow{U_3} \mathcal{C}_{T_1}(U_3 X, U_3 Y)$$

we apply U_3 to the T_2 bar construction for X . Using the assumption that T_3 and hence U_3 commutes with geometric realizations we have

$$\begin{aligned} U_3 |B_\bullet(F_{T_2}, T_2, U_2 X)| &= U_3 |B_\bullet(F_{T_3} F_{T_1}, U_1 U_3 F_{T_3} F_{T_1}, U_1 U_3 X)| \\ &= |B_\bullet(T_3 F_{T_1}, U_1 T_3 F_{T_1}, U_1 U_3 X)| \\ &\leftarrow |B_\bullet(F_{T_1}, U_1 F_{T_1}, U_1 U_3 X)| \end{aligned}$$

where the last map is induced by the unit map $\text{Id}_{\mathcal{C}_{T_1}} \rightarrow T_3$. □

We highlight two immediate corollaries of Theorem 1.1.

Corollary 4.1. The forgetful functor taking a non-empty $ho(\mathcal{C}_T)(X, Y)$ to $(ho\mathcal{C})_{hT}(X, Y)$ is surjective if and only if the differential d_r on $E_r^{0,0}$ is trivial for all $r \geq 2$.

Corollary 4.2. Suppose the portion of the spectral sequence computing $\pi_0 \mathcal{C}_T(X, Y)$ converges, i.e., there exists a base point ϵ and

$$\lim_s^1 \pi_1(\mathcal{C}_T^d(\text{sk}_s |B_\bullet \tilde{X}|, Y), \epsilon) = 0.$$

Then the forgetful functor taking $ho(\mathcal{C}_T)(X, Y)$ to $(ho\mathcal{C})_{hT}(X, Y)$ is injective if and only if $E_\infty^{t,t} = 0$ for $t > 0$.

Remark 4.3. As stated in [Bou89], every entry in the spectral sequence above should consist of based sets. We have chosen to omit the distinguished point $[\epsilon]$ in bidegree $(0,0)$ to simplify the statement of Theorem 1.1.

Remark 4.4. There are, in fact, a variety of obstruction sequences whose vanishing can give a lift of ϵ through the totalization tower. The following are special cases of [Bou89, §§2.4, 2.5, 5.2] for a cosimplicial object X_\bullet in a simplicial category \mathcal{D} :

- (1) The r th spectral sequence page $E_r^{p,q}$ is defined if there is an element $\epsilon_{r-1} \in \text{Tot}^{r-1} \mathcal{D}(X_\bullet, Y)$ which lifts to $\text{Tot}^{2r-2} \mathcal{D}(X_\bullet, Y)$, and the page depends naturally on ϵ_{r-1} .
- (2) Let $\epsilon_p \in \text{Tot}^p \mathcal{D}(X_\bullet, Y)$, and let ϵ_k be the projection of ϵ_p to $\text{Tot}^k \mathcal{D}(X_\bullet, Y)$. If

$$p/2 \leq k \leq p$$

then there is an obstruction element lying in $E_{p-k+1}^{p+1,p}$ which vanishes if and only if ϵ_k lifts to $\text{Tot}^{p+1} \mathcal{D}(X_\bullet, Y)$.

If Whitehead products vanish in each $\mathcal{D}(X_s, Y)$ (such as, e.g., when the mapping spaces of \mathcal{D} are H -spaces), then the range in which the obstruction classes are defined can be extended as follows:

- (1') The r th spectral sequence page $E_r^{p,q}$ is defined if there is an element $\epsilon_{r-2} \in \text{Tot}^{r-2} \mathcal{D}(X_\bullet, Y)$ which lifts to $\text{Tot}^{2r-3} \mathcal{D}(X_\bullet, Y)$, and the page depends naturally on ϵ_{r-2} .
- (2') Let $\epsilon_p \in \text{Tot}^p \mathcal{D}(X_\bullet, Y)$, and let ϵ_k be the projection of ϵ_p to $\text{Tot}^k \mathcal{D}(X_\bullet, Y)$. If

$$(p-1)/2 \leq k \leq p$$

then there is an obstruction element lying in $E_{p-k+1}^{p+1,p}$ which vanishes if and only if ϵ_k lifts to $\text{Tot}^{p+1} \mathcal{D}(X_\bullet, Y)$.

Taking $p = 1$ and $k = 0$ in (2') from Remark 4.4, we obtain the following useful refinement of Theorem 1.1:

Theorem 4.5 (Compare [GH05, Cor. 2.4.15]). Suppose T is a monad acting on a simplicial category \mathcal{C} and X and Y are T -algebras satisfying the conditions of Theorem 1.1. Moreover suppose that the derived mapping spaces $\mathcal{C}^d(T^n U \tilde{X}, UY)$ have the homotopy type of H -spaces. Then the T -algebra spectral sequence of Theorem 1.1 exists, its E_2 term is always defined, and there is a series of successively defined obstructions to realizing a map

$$[f] \in E_2^{0,0} = (ho\mathcal{C})_{hT}(UX, UY)$$

in the groups

$$E_2^{s+1,s} \cong \pi^{s+1} \pi_s(\mathcal{C}^d(T^\bullet U \tilde{X}, UY), f)$$

for $s \geq 1$. In particular, if these groups are all zero, then the map induced by the forgetful functor

$$ho(\mathcal{C}_T)(X, Y) \rightarrow (ho\mathcal{C})_{hT}(UX, UY)$$

is surjective. If, in addition

$$\pi^s \pi_s(\mathcal{C}^d(T^\bullet U \tilde{X}, UY), f) = 0$$

for all $s \geq 1$, then this map is a bijection.

We will primarily make use of this theorem in the following form:

Corollary 4.6. Suppose T is a monad on $Spectra$ and $X, Y \in Spectra_T$ satisfy the hypotheses of Theorem 1.1. Then the T -algebra spectral sequence of Theorem 1.1 exists, its E_2 term is always defined, and there is a series of successively defined obstructions to realizing a map

$$[f] \in E_2^{0,0} = (hoSpectra)_{hT}(UX, UY)$$

in the groups

$$\pi^{s+1} \pi_s(Spectra^d(T^\bullet U \tilde{X}, UY), f)$$

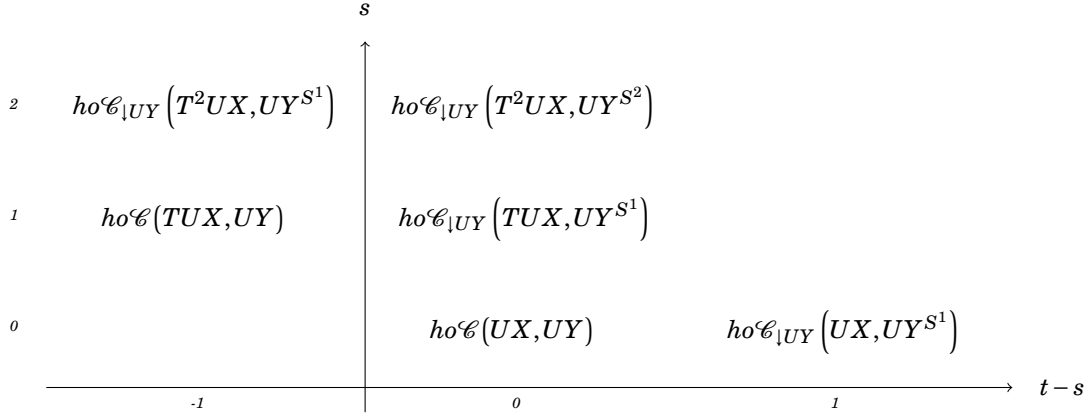


FIGURE 4.7. Low-degree terms on the E_1 page of the T -algebra spectral sequence, interpreted as homotopy classes of lifts.

for $s \geq 1$. In particular, if these groups are all zero, then the map induced by the forgetful functor

$$ho(Spectra_T)(X, Y) \rightarrow (hoSpectra)_{hT}(UX, UY)$$

is surjective. If, in addition

$$\pi^s \pi_s(Spectra^d(T^\bullet U\tilde{X}, UY), f) = 0$$

for all $s \geq 1$, then this map is a bijection.

4.2. Observations on E_1 . To simplify notation for this section we assume that the hypotheses of Theorem 1.1 are satisfied and we have replaced X with \tilde{X} and Y with its fibrant replacement if necessary.

Provided all of the terms in $E_1^{s,t}$ of the T -algebra spectral sequence for $t > 0$ are abelian groups, then we can avoid using the normalized cocomplex in [Bou89] and instead use Moore cochains. This happens if, for example, the mapping spaces $\mathcal{C}(T^n UX, UY)$ have the homotopy type of H -spaces. We then have a reinterpretation of the terms in $E_1^{s,t}$, via the tensor-cotensor adjunction:

$$E_1^{s,t} = \pi_t(\mathcal{C}(T^s UX, UY), \varepsilon) \cong \pi_0 \mathcal{C}_{|UY}(T^s UX, UY^{S^t}).$$

This displays $E_1^{s,t}$ as a set of homotopy classes of lifts in the diagram below, with homotopies fiberwise over ε :

$$\begin{array}{ccc} & & UY^{S^t} \\ & \nearrow \text{dashed} & \downarrow \\ T^s UX & \xrightarrow{\varepsilon} & UY \end{array}$$

As in the proof of Theorem 1.1, $T^\bullet UX$ is Reedy cofibrant and it follows that each $T^n UX$ is cofibrant in \mathcal{C} . Since U creates cotensors, preserves fibrations, and Y is fibrant in \mathcal{C}_T , we see UY^{S^t} is fibrant in \mathcal{C} . Now the overcategory is a cofibrantly generated simplicial model category whose cofibrations/fibrations/weak equivalences are those of \mathcal{C} . So these objects are cofibrant and fibrant respectively in $\mathcal{C}_{|UY}$. Regarding UX as an object over UY by a chosen map $\varepsilon: UX \rightarrow UY$, we now obtain the identification

$$E_1^{s,t} \cong ho\mathcal{C}_{|UY}(T^s UX, UY^{S^t}) \quad \text{for } t > 0.$$

4.3. Applicable contexts. This section will be devoted to demonstrating that the hypotheses of Theorem 1.1 are satisfied in many categories of interest.

4.3.1. Simplicial algebraic theories.

Theorem 4.8. If T is a monad on $\mathcal{S}et^S$ associated to an S -graded algebraic theory as in Section 2.2, then the T -algebra spectral sequence of Theorem 1.1 can be applied to any $X, Y \in \mathcal{S}et_T^S$.

Proof. By Proposition 3.4 we see that T is Quillen. Remark 3.11 shows that T commutes with geometric realizations. Finally Proposition 3.21 shows that any X is resolvable. \square

For example, by Remark 2.18 we can apply the T -algebra spectral sequence to analyze spaces of operad maps. Since the space of operad maps from an operad \mathcal{O} to the endomorphism operad of an object X (when defined) is in correspondence with the space of algebra structures on X [Rez96], one can, in theory, use this spectral sequence to analyze algebra structures on X .

4.3.2. G -actions.

For the following result one can use any of the standard cofibrantly generated models for the category of spectra which is enriched in spaces and such that the tensor product of a subcellular inclusion of spaces with a cellular spectrum is naturally a subcellular inclusion of spectra.

Proposition 4.9. Let G be a topological group admitting a cellular structure such that the unit is the inclusion of a sub-complex. Let $TX = G_+ \wedge X$ be the monad on based spaces/spectra whose algebras are G -spaces/spectra. Then the T -algebra spectral sequence of Theorem 1.1 can be applied to any X, Y in these categories.

Proof. It is well known and straightforward to show using Theorem 3.2 and Remark 3.3 that T is Quillen. Since geometric realization commutes with smash products in either of these categories we see that T commutes with geometric realization. Since the unit transformation applied to cellular spectra gives an inclusion of subcomplexes, by Proposition 3.23 we see that every X is resolvable. \square

In the case of G -spaces or G -spectra the T -algebra spectral sequence of Theorem 1.1 takes a familiar form. The bar resolution applied to X is the standard cofibrant replacement $EG_+ \wedge X \rightarrow X$ in the naive model structure. The skeletal filtration on the bar resolution corresponds to the bar filtration on EG and our spectral sequence computing the homotopy groups of the space of G -maps between X and Y becomes the homotopy fixed point spectral sequences computing the homotopy groups of $F(X, Y)^{hG}$ where $F(X, Y)$ is the corresponding G -space of maps.

Remark 4.10. As expected, the homotopy G -spaces/spectra (i.e., the homotopy T -algebras for T as above) will correspond to those spaces/spectra which admit a G -action in the homotopy category. Morphisms of homotopy G -spaces/spectra are maps in the homotopy category which commute with the G -action. In particular, any G -map which is non-equivariantly null-homotopic is necessarily trivial in the category of homotopy G -spaces (see Section 5.1).

4.3.3. Algebras over operads.

Proposition 4.11. Let T be a monad acting on a symmetric monoidal model category \mathcal{C} , and suppose that

- (a) \mathcal{C} satisfies the conditions of Proposition 2.19,
- (b) geometric realization commutes with the symmetric monoidal structure on \mathcal{C} ,
- (c) T arises from a cofibrant admissible operad (see [BM03]),
- (d) and one of the following conditions holds:
 - (i) The underlying category \mathcal{C} is $\mathcal{S}et^S$ for some set S .

- (ii) Relative cell complexes in \mathcal{C} are monomorphisms and for every T -algebra Y which is cellular in \mathcal{C} , the unit map $Y \rightarrow TY$ is a cellular inclusion.
- (iii) For every T -algebra Y which is cofibrant in \mathcal{C} , the unit map $Y \rightarrow TY$ is the cofibrant inclusion of a summand and \mathcal{C} is pointed.

Then the T -algebra spectral sequence of Theorem 1.1 can be applied to any $X, Y \in \mathcal{C}_T$.

Proof. The hypotheses of Proposition 2.19 ensure that \mathcal{C}_T is a bicomplete simplicial category. The definition of admissibility [BM03, §4] is, essentially, that Theorem 3.2 can be applied to show that T is Quillen. Remark 3.11 and the assumption that geometric realization commutes with the monoidal structure shows that T commutes with geometric realization.

Since our operad is cofibrant we can replace any T -algebra by one which is cellular or cofibrant in \mathcal{C} by [BM03, Thm. 3.5 (b)]. Finally by the remaining hypothesis we can apply either Proposition 3.21, Proposition 3.23, or Proposition 3.25 to see that any T -algebra is resolvable. \square

Proposition 4.12. Let T be a monad acting on a pointed symmetric monoidal model category \mathcal{C} , and suppose that

- (a) \mathcal{C} satisfies the conditions of Proposition 2.19,
- (b) geometric realization commutes with the symmetric monoidal structure on \mathcal{C} ,
- (c) and T arises from an admissible operad $W\mathcal{O}$, where $W\mathcal{O}$ is the Boardman-Vogt cofibrant replacement of an operad \mathcal{O} (see [BM06]) such that $\mathcal{O}(0) = \mathcal{O}(1) = *$.

Then the T -algebra spectral sequence of Theorem 1.1 can be applied to any $X, Y \in \mathcal{C}_T$.

Proof. We will apply Proposition 4.11 using the hypotheses that \mathcal{C} is pointed and that the unit map is the inclusion of a summand. As shown in [BM06] the Boardman-Vogt construction yields a functorial cofibrant replacement of our operad. By Lemma 4.13 $(W\mathcal{O})(1) = *$, so the unit map $X \rightarrow TX$ is always the inclusion of a summand.

Since $W\mathcal{O}$ is cofibrant, replacing X with a cofibrant replacement if necessary, we can assume X is cofibrant in \mathcal{C} by [BM03, Thm. 3.5 (b)]. Since \mathcal{C} is a symmetric monoidal model category it is straightforward to apply the pushout-product axiom and induction on n to see that $X^{\otimes n}$ is cofibrant. Finally, since our cofibrant operad is Σ -cofibrant [BM06, §2.4] we see that $W\mathcal{O}(n) \otimes X^{\otimes n}$ is a retract of a cellular complex built with free Σ_n -cells. It follows that $W\mathcal{O}(n) \otimes_{\Sigma_n} X^{\otimes n}$ is cofibrant which in turn implies TX is cofibrant. \square

Lemma 4.13. Suppose that \mathcal{O} is an operad such that $\mathcal{O}(0) = \mathcal{O}(1) = *$. Then $W\mathcal{O}(1) = *$.

Proof. The result follows immediately from the construction of $W\mathcal{O}$ in [BM06] and we use the notation therein. Namely, under the given hypotheses all of the maps in the sequential colimit

$$W(H, \mathcal{O})(n) = \operatorname{colim} (\mathcal{O}(n) = W_0(H, \mathcal{O})(n) \rightarrow W_1(H, \mathcal{O})(n) \rightarrow \dots)$$

are isomorphisms when $n = 1$ (H is the unit interval here). To see this, one observes that the right-hand (and therefore left-hand) vertical maps in the pushout [BM06, (13)] are isomorphisms for $n = 1$: For trees G with a single input edge, the objects $\underline{\mathcal{O}}(G)$ and $\underline{\mathcal{O}}^-(G)$ are equal (all vertices of G are univalent, and if $\mathcal{O}(1) = *$ then $\underline{\mathcal{O}}_c(G) = \underline{\mathcal{O}}(G)$ for any subset of univalent vertices c). As an aside, note that this implies the vertical arrows in the pushout diagram at the end of [BM06, §3] are isomorphisms for $n = 1$, and hence $\mathcal{F}_*(\mathcal{O})(1) = \mathcal{O}(1) = *$. Moreover, this implies $(H \otimes \mathcal{O})^-(G) = H(G) \otimes \mathcal{O}^-(G)$. Therefore the vertical maps in [BM06, (13)] are isomorphisms and $W(H, \mathcal{O})(1) = W_0(H, \mathcal{O})(1) = \mathcal{O}(1)$. \square

Let R be a commutative ring spectrum. For the following corollary one can use any symmetric monoidal category of R -modules satisfying hypotheses (a) and (b) of Proposition 4.12. These conditions are easily verified in the standard cases such as those of [EKMM97, HSS00, MM02].

Corollary 4.14. Suppose T is the monad associated to the Boardman-Vogt replacement of either the associative or the commutative operad (so it is an A_∞ or E_∞ operad) acting on $R\text{-Mod}$. Then the T -algebra spectral sequence of Theorem 1.1 can be applied to any T -algebras X and Y .

5. COMPUTATIONS

5.1. G -actions. The next two examples provide, respectively, an example of a non-trivial G -map which is trivial as a homotopy G -map and an example of a non-trivial homotopy G -map which does not lift to a G -map.

Example 5.1. Regard \mathbb{R} as a C_2 -space via the sign action. Then applying one point compactification to the inclusion

$$\{0\} \rightarrow \mathbb{R}$$

yields an essential map

$$e_\sigma: S^0 \rightarrow S^\sigma$$

of pointed C_2 -spaces.

Taking the trivial map as our base point of $\text{Top}_{C_2}(S^0, S^\sigma)$ and applying the spectral sequence of Theorem 1.1 we have ¹:

$$E_2^{s,t} = H^s(C_2; \pi_t S^\sigma) \Rightarrow \pi_{t-s}(S^\sigma)^{hC_2}.$$

As noted in Remark 4.10, e_σ must represent the trivial map in the category of homotopy C_2 -spaces. The spectral sequence confirms this since $E_2^{0,0} = (\pi_0 S^\sigma)^{C_2} = 0$. In fact, since the homotopy groups of S^1 are concentrated in degree 1, this spectral sequence is concentrated on the line $t = 1$ and necessarily collapses at E_2 . Computing the group cohomology with twisted coefficients we see that the only non-zero contribution is from $E_2^{1,1} = \mathbb{Z}/2$ which detects the map e_σ above.

$$\begin{array}{ccc} & s & \\ & \bullet & \\ 0 & \varepsilon & \\ & 0 & t-s \end{array}$$

FIGURE 5.2. T -algebra spectral sequence for C_2 -equivariant maps $S^0 \rightarrow S^\sigma$.

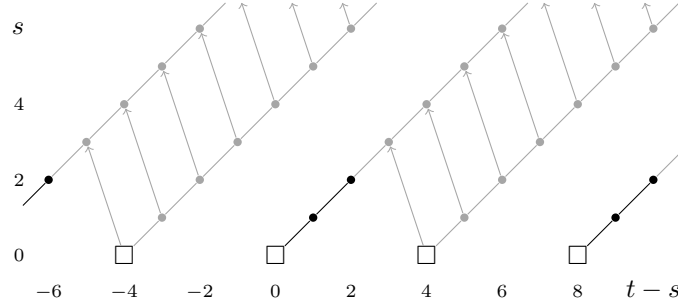
Example 5.3. Let C_2 act on KU via complex conjugation. The C_2 -action on $\pi_* KU$ is trivial precisely on those homotopy groups generated by even powers of the Bott map. In particular, if we regard S^4 as having a trivial C_2 action we obtain a non-trivial map

$$\beta^2: S^4 \rightarrow KU$$

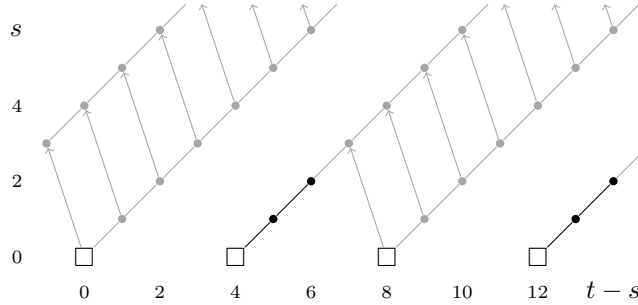
in the category of homotopy C_2 -spectra.

The spectral sequence of Theorem 1.1 computing the homotopy groups of the space of C_2 -equivariant maps from S^0 to KU is (the connective cover of) the homotopy fixed point spectral sequence. After 2-completion this spectral sequence converges to the homotopy of KO and there is a well-known differential $d_3(\beta^2) = \eta^3$, which follows from a comparison with the Adams-Novikov spectral sequence and the relation $\eta^4 = 0$ in $\pi_* S$.

¹Normally instability, e.g., actions of the fundamental group, prevents getting such a simple description of the E_2 term, however in this case S^σ is non-equivariantly an Eilenberg-MacLane space for \mathbb{Z} and so the second half of the refinements in Remark 4.4 apply.

FIGURE 5.4. Homotopy fixed-point spectral sequence for $\pi_*(KU^{hC_2})$.

Since the T -algebra spectral sequence computing $\pi_* \text{Spectra}_{C_2}(S^4, KU)$ is just a shift of this spectral sequence we see that the element $\beta^2 \in E_2^{0,0}$ supports a d_3 and does not lift to a map of C_2 -spectra.

FIGURE 5.5. T -algebra spectral sequence for C_2 -equivariant maps $S^4 \rightarrow KU$.

5.2. Methodology for calculating the E_2 -term. The purpose of this section is to prove Theorem 5.6, which will give criteria for obtaining a computationally useful, i.e. algebraic, description of the E_2 term from Theorem 4.5.

Theorem 5.6. Suppose that T is a monad acting on \mathcal{C} and $X, Y \in \mathcal{C}_T$ satisfy the hypotheses of Theorem 4.5. By passing to weakly equivalent replacements if necessary, assume that Y is fibrant and $X = \tilde{X}$ as in Definition 3.18.

Suppose there is a functor

$$\pi_* : ho\mathcal{C} \rightarrow \mathcal{D}$$

such that

- (a) The associated map

$$\pi_* : ho\mathcal{C}(T^s UX, UY^{S^t}) \rightarrow \mathcal{D}(\pi_* T^s UX, \pi_* UY^{S^t})$$

is an isomorphism for all $s, t \geq 0$.

- (b) There is a natural isomorphism $\pi_* TX \cong T_{\text{alg}} \pi_* X$ for a monad T_{alg} compatible with the structure homomorphisms of T and T_{alg} .
- (c) The categories \mathcal{D} and $\mathcal{D}_{T_{\text{alg}}}$ are categories of *Set*-valued models for some graded algebraic theories (i.e., they are algebraic categories in the sense of Quillen).
- (d) For $t \geq 1$, $\pi_* Y^{S^t}$ is naturally an abelian group object in the category of T_{alg} -algebras over $\pi_* Y$.

Then the E_2 term of the T -algebra spectral sequence exists and can be identified as follows:

$$\begin{aligned} E_2^{0,0} &\cong \mathcal{D}_{T_{\text{alg}}}(\pi_* UX, \pi_* UY) \\ E_2^{s,t} &\cong H_{AQ, \pi_* UY}^s(\pi_* UX; \pi_* UY^{S^t}) \quad \text{for } t > 0. \end{aligned}$$

Where the cohomology groups on the second line are the associated André-Quillen cohomology groups of our T_{alg} -algebra $\pi_* X$ viewed as an algebra over $\pi_* Y$ via a choice of an element in $E_2^{0,0}$ as in Definition 3.8.

Proof. First we identify $E_2^{0,0}$: As remarked in the proof of Theorem 1.1, this is computed by the equalizer

$$E_2^{0,0} = \text{eq}(ho\mathcal{C}(UX, UY) \rightrightarrows ho\mathcal{C}(TUX, UY))$$

where the morphisms are induced by the T -algebra structures on X and Y respectively. Using the isomorphisms from condition (a) we obtain the identification

$$E_2^{0,0} \cong \text{eq}(\mathcal{D}(\pi_* UX, \pi_* UY) \rightrightarrows \mathcal{D}(\pi_* TUX, \pi_* UY)).$$

Applying π_* and using condition (b) we obtain

$$E_2^{0,0} \cong \mathcal{D}_{T_{\text{alg}}}(\pi_* UX, \pi_* UY) \cong \text{eq}(\mathcal{D}(\pi_* UX, \pi_* UY) \rightrightarrows \mathcal{D}(T_{\text{alg}} \pi_* UX, \pi_* UY)).$$

Now to identify the remainder of the E_2 term we pick a map $f: UX \rightarrow UY$ representing some element $[f] \in E_2^{0,0}$. By Theorem 1.1 for $t > 0$

$$E_2^{s,t} \cong \pi^s \pi_t(\mathcal{C}(T^s UX, UY), f).$$

Since the hypotheses of Theorem 4.5 are satisfied

$$\pi_t(\mathcal{C}(T^s UX, UY), f)$$

is a cosimplicial abelian group for $t > 0$ and the cohomotopy group π^s can be calculated as the s th cohomology group of the associated Moore cochain complex. By the discussion in Section 4.2 we see

$$\pi_t(\mathcal{C}(T^s UX, UY), f) \cong ho\mathcal{C}_{|UY}(T^s UX, UY^{S^t}).$$

Applying the homotopy invariant functor π_* and using conditions (a) and (b) we obtain

$$\begin{aligned} E_2^{s,t} &\cong H^s(\mathcal{D}_{\downarrow \pi_* Y}(T_{\text{alg}}^s \pi_* UX, \pi_* UY^{S^t})) \\ &\cong H^s(\mathcal{D}_{T_{\text{alg}} \downarrow \pi_* Y}(F_{T_{\text{alg}}} T_{\text{alg}}^s \pi_* UX, \pi_* UY^{S^t})) \end{aligned}$$

where the last isomorphism follows because $\pi_* UY^{S^t}$ is an algebra over T_{alg} . These cohomology groups are, by definition, the *cotriple cohomology groups* of $\pi_* UX$ with respect to the cotriple associated to the monad T_{alg} .

To complete the proof we must identify the cotriple cohomology with André-Quillen cohomology; this follows from [Qui69, II.5.Theorem 5]. □

5.3. Algebras over an operad in spectra. For one example of applying Theorem 5.6 let k be a field and let T be the monad

$$X \mapsto TX = \bigvee_{n \geq 0} K_n \otimes X^{\wedge_{Hk} n}$$

on Hk -module spectra associated to the A_∞ operad. In Section 5.5 we will verify the conditions (a) and (b) from Theorem 5.6 where

$$T_{\text{alg}} \pi_* X \cong \bigoplus_{n \geq 0} (\pi_* X)^{\otimes_k n}$$

is a monad on graded k -modules whose algebras are graded associative k -algebras.

Of course graded k -modules and graded associative k -algebras satisfy condition (c). In the category of associative algebras over $\pi_* Y$, the abelian group objects are the square-zero extensions of $\pi_* Y$ such as $\pi_* Y^{S^t} \cong \pi_* Y \oplus \pi_* \Sigma^{-t} Y$ and hence condition (d) from Theorem 5.6 is satisfied.

So we obtain a spectral sequence

$$E_2^{s,t} \Rightarrow \pi_{t-s} A_\infty Hk\text{-}\mathcal{A}lg(X, Y)$$

such that

$$E_2^{0,0} = k\text{-}\mathcal{A}lg(\pi_* X, \pi_* Y)$$

and

$$E_2^{s,t} = H_{AQ, \pi_* Y}^s(\pi_* X, \pi_* Y^{S^t}) \quad \text{for } t > 0,$$

where the cohomology groups are calculated in the category of graded associative k -algebras over $\pi_* Y$. For $s = 0$ these can be identified with the derivations of $\pi_* X$ into $\pi_{*+t} Y$ and for $s > 0$ these are the $s + 1$ st Hochschild cohomology groups

$$HH^{s+1}(\pi_* X; \pi_{*+t} Y) \cong \text{Ext}_{\pi_* X \otimes_k (\pi_* X)^{\text{op}}}^{s+1}(\pi_* X, \pi_{*+t} Y)$$

of $\pi_* X$ with coefficients in $\pi_{*+t} Y$ [Qui70, Prop. 3.6].

As shown in Section 5.5, we obtain a similar result where T is the monad on Hk -modules whose algebras are the E_∞ -algebras in this category. In that case T_{alg} is the monad on graded k -modules whose algebras are Dyer-Lashof algebras [BMMS86]. If k is a field of characteristic 0 these are just the graded commutative k -algebras.

Example 5.7. In the category of Hk -modules, consider the A_∞ -algebra $Hk \wedge \Sigma_+^\infty \Omega SU(n+1)$ for each $n \geq 1$. The homotopy of this algebra is a polynomial algebra $R = R_n$ on generators $\{x_i\}_{1 \leq i \leq n}$ where the $|x_i| = 2i$. To compute the A_∞ self-maps we apply our spectral sequence and the discussion above to compute

$$\begin{aligned} E_2^{0,0} &\cong k\text{-}\mathcal{A}lg(R, R) \cong \prod_{1 \leq i \leq n} (R)_{2i} \\ E_2^{0,t} &\cong \text{Der}(R; \Sigma^{-t} R) \cong \prod_{1 \leq i \leq n} (R)_{2i+t} \\ E_2^{s,t} &\cong \text{Ext}_{R \otimes_k R^{\text{op}}}^{s+1}(R, \Sigma^{-t} R) \quad \text{for } s > 0. \end{aligned}$$

In particular, these groups are zero for t odd, hence $E_{2i} = E_{2i+1}$. The Hochschild cohomology groups can be calculated by first pulling back the $R \otimes_k R^{\text{op}}$ action to an $R \otimes_k R$ action via the isomorphism defined by

$$x_i \otimes 1 \mapsto x_i \otimes 1, \quad 1 \otimes x_i \mapsto x_i \otimes 1 - 1 \otimes x_i.$$

Since the second copy of R acts trivially on the source we obtain an $(R \otimes_k R)$ -free resolution of R by applying $R \otimes_k -$ to the Koszul resolution of k :

$$(\Lambda_k[\sigma x_1, \dots, \sigma x_n] \otimes_k R \rightarrow k) \xrightarrow{R \otimes_k -} (R \otimes_k \Lambda_k[\sigma x_1, \dots, \sigma x_n] \otimes_k R \rightarrow R)$$

Here σx_i has bidegree $(1, 2i)$ and $d(\sigma x_i) = x_i$. Using this resolution we see that

$$\text{Ext}_{R \otimes_k R^{\text{op}}}^*(R, R) \cong (\Lambda_k[\sigma x_1, \dots, \sigma x_n])^* \otimes_k R.$$

So the Hochschild cohomology groups vanish above cohomological degree n and hence the T -algebra spectral sequence is concentrated on the first $n - 1$ lines and must collapse at E_n for $n \geq 2$. In particular, if $n = 1$ then the spectral sequence collapses at E_2 onto the 0-line.

Hence there are no obstructions to lifting a map of k -algebras

$$H_* \Omega SU(n+1) \rightarrow H_* \Omega SU(n+1)$$

to a map of A_∞ -algebras if $n \leq 3$ and such a map is homotopically unique if $n \leq 2$. For $n = 1$ this result is expected since $\Omega SU(2) \cong \Omega \Sigma S^2$ is stably a free A_∞ -algebra.

The previous computation did not depend on the A_∞ algebra $Hk \wedge \Sigma_+^\infty \Omega SU(n+1)$ so much as the fact that its ring of homotopy groups is polynomial on generators in even degrees. In particular, we have the following:

Proposition 5.8. Let R_n be a polynomial algebra on n generators in even degrees. Then for n less than 4, there is a unique Hk -algebra V up to homotopy such that $\pi_* V \cong R_n$. In particular, all such algebras are weakly equivalent to the commutative Hk -algebra HR_n .

Example 5.9. If we allow n to go to infinity in Example 5.7 then $\Omega SU \simeq BU$ is an infinite loop space and consequently $Hk \wedge \Sigma_+^\infty \Omega SU$ is an E_∞ -algebra in Hk -modules.

When the characteristic of k is zero, the results in Section 5.5 enable us to compute the space of E_∞ self maps. We have the following identification of the E_2 -term, where

$$R = H_*(\Omega SU) \cong k[x_i]_{i \geq 1}$$

and $k\text{-}\mathcal{A}lg$ is the category of commutative k -algebras:

$$\begin{aligned} E_2^{0,0} &\cong k\text{-}\mathcal{A}lg(R, R) \cong \prod_{i \geq 1} (R)_{2i} \\ E_2^{0,t} &\cong \text{Der}(R, \Sigma^{-t} R) \cong \prod_{i \geq 1} (R)_{2i+t} \\ E_2^{s,t} &\cong H_{AQ,R}^s(R; \Sigma^{-t} R) \quad \text{for } t > 0. \end{aligned}$$

Since R is a polynomial algebra, it is smooth and by Proposition 5.13 all higher André-Quillen cohomology groups vanish. As a consequence we see the spectral sequence collapses at E_2 onto the 0 line. Hence every map of homology rings lifts to a homotopically unique map of E_∞ -algebras in Hk -modules.

Alternatively one could deduce the conclusion from the previous example in a more direct fashion. There is a map $\mathbb{C}P^\infty \rightarrow BU$ which maps the reduced homology of $\mathbb{C}P^\infty$ isomorphically onto indecomposable generators for the homology of BU . Since BU is a based E_∞ space this canonically extends to a map of E_∞ -algebras

$$T\mathbb{C}P^\infty \rightarrow \Sigma_+^\infty BU.$$

Localizing rationally this map is an equivalence and as a consequence $\Omega SU \simeq BU$ is rationally a free E_∞ spectrum. By the following example the spectral sequence of Theorem 1.1 always collapses at the E_2 term onto the 0 line when the source is such a spectrum. We see that $Hk \wedge \Omega SU$ is equivalent to a free E_∞ -algebra in Hk -modules so the spectral sequence in Example 5.9 collapses.

Since rational localization is smashing, the extension functor from E_∞ algebras to E_∞ algebras in $H\mathbb{Q}$ modules is an equivalence for every rational E_∞ ring spectrum. From this we obtain for any rational E_∞ ring spectra X and Y

$$E_\infty(X, Y) \simeq E_\infty H\mathbb{Q}\text{-}\mathcal{A}lg(H\mathbb{Q} \wedge X, Y) \simeq E_\infty H\mathbb{Q}\text{-}\mathcal{A}lg(X, Y).$$

So there is no difference homotopically between the space of E_∞ maps between two rational E_∞ rings and the space of E_∞ -algebra maps in $H\mathbb{Q}$ -modules.

Example 5.10. If $X = TM$ is a free E_∞ ring spectrum then the unit map $X \rightarrow TM$ is a map of E_∞ ring spectra and defines a section of the bar resolution. Consequently the spectral sequence of Theorem 1.1 computing the homotopy of $E_\infty(X, Y)$ collapses at E_2 onto the 0 line. So in this case the edge homomorphism $\pi_0 E_\infty(X, Y) \rightarrow H_\infty(X, Y)$ is an isomorphism. Moreover there is a homotopy equivalence $E_\infty(X, Y) \simeq \text{Spectra}(M, Y)$.

Example 5.11. We will now construct infinitely many E_∞ maps that all induce the same H_∞ -map. For a space X , recall that the cotensor $H\mathbb{Q}^X$ is an E_∞ ring spectrum satisfying $\pi_* H\mathbb{Q}^X \cong H^{-*}(X)$. Now to calculate the homotopy groups of $E_\infty(H\mathbb{Q}^{S^2}, H\mathbb{Q}^{S^3})$ we apply the spectral sequence from

Theorem 1.1 and the identification of the E_2 -term above. As a base point we will take a ‘trivial’ map ϵ of E_∞ -rings induced by a map $S^3 \rightarrow * \rightarrow S^2$.

To calculate the E_2 term we have

$$E_2^{0,0} \cong \mathbb{Q}\text{-}\mathcal{CAlg}(\pi_* H\mathbb{Q}^{S^2}, \pi_* H\mathbb{Q}^{S^3}) \cong \text{Ind}_{-3}(\pi_* H\mathbb{Q}^{S^2}) = 0 = \epsilon.$$

For $t > 0$ we use the map ϵ above to regard $\pi_* H\mathbb{Q}^{S^2}$ as a commutative algebra over $\pi_* H\mathbb{Q}^{S^3}$ and obtain

$$E_2^{s,t} \cong H_{AQ}^s(\pi_* H\mathbb{Q}^{S^2}; \pi_{*+t} H\mathbb{Q}^{S^3}) \cong \mathbb{Q}\text{-}\mathcal{CAlg}_{\pi_* H\mathbb{Q}^{S^3}}^d(\pi_* H\mathbb{Q}^{S^2}, \pi_* H\mathbb{Q}^{S^3} \oplus \Sigma_{\mathbb{Q}}^s \pi_{*+t} H\mathbb{Q}^{S^3})$$

where the right-hand side is the derived homomorphisms of simplicial commutative \mathbb{Q} -algebras over the constant simplicial algebra $\pi_* H\mathbb{Q}^{S^3}$ into the square-zero extension of this algebra by the s th suspension of $\pi_{*+t} H\mathbb{Q}^{S^3}$ in simplicial $\pi_* H\mathbb{Q}^{S^3}$ -modules. To calculate these derived homomorphisms we first construct a cofibrant replacement of the source.

We construct this cofibrant replacement of the exterior algebra $\pi_* H\mathbb{Q}^{S^2}$ via a homotopy pushout diagram of cellular algebras. Let e_{-2} denote a generator of a one-dimensional \mathbb{Q} -module in dimension -2 and $T_{\text{alg}}(e_{-2})$ the free simplicial commutative \mathbb{Q} -algebra on this module. A non-zero map

$$e_{-2} \rightarrow \pi_{-2} H\mathbb{Q}^{S^2}$$

canonically extends to a map of simplicial commutative \mathbb{Q} -algebras

$$T_{\text{alg}}(e_{-2}) \rightarrow \pi_* H\mathbb{Q}^{S^2}.$$

Both of these algebras are constant and the kernel of this surjective map is (e_{-2}^2) . We can similarly construct a map

$$T_{\text{alg}}(f_{-4}) \rightarrow T_{\text{alg}}(e_{-4})$$

which surjects onto this kernel. Let $C(f_{-4})$ denote a cone on the one dimensional \mathbb{Q} -module in degree -4 , which sits in a factorization of the 0 map

$$\mathbb{Q}\{f_{-4}\} \rightarrow C(f_{-4}) \rightarrow 0$$

by a cofibration followed by an acyclic fibration in simplicial graded \mathbb{Q} -modules. Since

$$f_{-4} \mapsto 0 \in \pi_* H\mathbb{Q}^{S^2}$$

we obtain a map of simplicial \mathbb{Q} -modules $C(f_{-4}) \rightarrow \pi_* H\mathbb{Q}^{S^2}$.

We now obtain a homotopy pushout diagram

$$\begin{array}{ccc} T_{\text{alg}}(f_{-4}) & \xrightarrow{e_{-2}^2} & T_{\text{alg}}(e_{-2}) \\ \downarrow & & \downarrow \\ T_{\text{alg}}(Cf_{-4}) & \longrightarrow & T_{\text{alg}}(Cf_{-4}) \otimes_{T_{\text{alg}}(f_{-4})} T_{\text{alg}}(e_{-2}) \end{array}$$

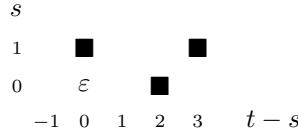
Now we have an induced map from this pushout to $\pi_* H\mathbb{Q}^{S^2}$ which an easy computation shows is a quasi-isomorphism.

We can now map out of this pushout diagram into the square-zero extension $\pi_* H\mathbb{Q}^{S^3} \oplus \pi_{*+t} H\mathbb{Q}^{S^3}$ and obtain a long exact sequence. Note since

$$\mathbb{Q}\text{-}\mathcal{CAlg}_B^d(TM, B \oplus \Sigma_{\mathbb{Q}}^s N) \cong B\text{-}\mathcal{Mod}(B \otimes M, \Sigma_{\mathbb{Q}}^s N)$$

this is a long exact sequence of Ext groups.

From this long exact sequence we now obtain the following E_2 -term:

FIGURE 5.12. T -algebra spectral sequence for E_∞ maps $HQ^{S^2} \rightarrow HQ^{S^3}$.

All other entries are trivial so the spectral sequence collapses at E_2 . The \mathbb{Q} in $E_2^{1,1}$ detects an infinite family of E_∞ maps which, because they land in positive filtration, induce the same H_∞ map ϵ . It can be shown that this infinite family is generated by the morphism of E_∞ rings induced by the Hopf map $S^3 \rightarrow S^2$.

In the previous example, the spectral sequence vanished above the 1-line guaranteeing collapse of the spectral sequence and an algebraic description of the space of E_∞ -maps. This is because the map $\mathbb{Q} \rightarrow \pi_* HQ^{S^2}$ is a local complete intersection morphism and hence the higher André-Quillen cohomology groups vanish. We will say a morphism of $A \rightarrow B$ of graded commutative rings is a local complete intersection, resp. smooth, resp. étale, if the relative cotangent complex $L_{B/A}$ [Qui70] has projective dimension at most one, resp. is projective, resp. is contractible.

Proposition 5.13. Suppose $f: k \rightarrow R$ and $k \rightarrow S$ are morphisms of rational E_∞ -rings. Suppose the spectral sequence of Theorem 1.1 computing the space of E_∞ -ring maps under k between R and S has a well-defined E_2 -term (e.g., there is a map to serve as the base point). Then if the morphism f on homotopy groups is

- (1) *a local complete intersection* then the spectral sequence collapses at the E_2 page onto the 0 and 1 lines and every H_∞ map can be realized by an E_∞ -map, although possibly non-uniquely.
- (2) *smooth* then the spectral sequence collapses at the E_2 page onto the 0 line and every H_∞ map can be realized, uniquely up to homotopy, by an E_∞ -map.
- (3) *étale* then the spectral sequence collapses at the E_2 page and $E_2^{s,t} = 0$ if $t > 0$. As a consequence, the mapping space is homotopically discrete and every H_∞ map can be realized, up to a contractible space of choices, by an E_∞ -map.

Proof. All of the results follow from the vanishing of the relevant André-Quillen cohomology groups [Qui70, Thm. 2.4 (ii)] and our identification of $E_2^{0,0}$ with the set of H_∞ maps. \square

Example 5.14. We now construct examples of H_∞ ring maps that do not lift to E_∞ ring maps. The argument below does not make explicit use of the spectral sequence beyond the identification of the H_∞ maps, although it does have consequences for its behavior.

Let M be the Heisenberg 3-manifold: the quotient of the group of uni-upper triangular 3×3 real matrices by the subgroup with all integer entries. Since M is a quotient of a contractible group by a discrete subgroup it is a $K(\pi, 1)$. The commutator subgroup of π is free abelian of rank one and π fits into the short exact sequence of groups

$$1 \rightarrow \mathbb{Z} \rightarrow \pi \rightarrow \mathbb{Z} \times \mathbb{Z} \rightarrow 1.$$

In particular M is a nilpotent space.

Applying the classifying space functor to the above exact sequence we see that up to homotopy, M can also be realized as the total space of an S^1 bundle over the torus T^2 . This S^1 bundle is classified by the generator of $\mathbb{Z} \cong H^2(T^2; \mathbb{Z}) \cong [T^2, BS^1]$.

A computation with the Serre spectral sequence shows $\pi_* HQ^M$ is generated by exterior classes x and y in degree -1, polynomial classes α and β in degree -2, and satisfy

$$xy = \alpha^2 = \beta^2 = \alpha\beta = x\alpha = y\beta = y\alpha = 0$$

As a consequence we see:

$$(5.15) \quad H_\infty(H\mathbb{Q}^M, H\mathbb{Q}^{S^2}) \cong E_2^{0,0} \cong \text{Ind}_{-2}(\pi_* H\mathbb{Q}^M) = \mathbb{Q}\{\alpha, \beta\}.$$

There are also Massey product relations $\alpha \in \langle x, x, y \rangle$ and $\beta \in \langle y, y, x \rangle$ with no indeterminacy.

Any map from $H\mathbb{Q}^M$ to $H\mathbb{Q}^{S^2}$ sends x and y to zero for degree reasons. Now α and β are Massey products in x and y and Massey products in $H\mathbb{Q}^*M$ correspond to Toda brackets in $H\mathbb{Q}^M$. Since E_∞ maps preserve Toda brackets, they must also send α and β to zero. So α and β must support differentials and correspond to H_∞ -maps which do not lift to E_∞ maps.

5.4. Coker J and maps of E_∞ ring spectra. The following example is a joint result of the second named author and Nick Kuhn.

For this example we will need to recall the definitions of some classical infinite loop spaces/connective spectra (cf. [HS78, p.271]). Let $SL_1S^0 = GL_1S^0\langle 0 \rangle$ denote the 1-component of QS^0 . At an odd prime p let q be an integer generating $(\mathbb{Z}/p^2)^\times$, the choice does not matter. Define J to be the fiber of the map

$$BU^\otimes \xrightarrow{\psi^q/\psi^1} BU^\otimes$$

where BU^\otimes is the 1-component of p -local K -theory and ψ^q is the q th Adams operation. The d -invariant defines a map $S^0 \rightarrow KU$ which restricts to a map $SL_1S^0 \xrightarrow{D} BU^\otimes$ which in turn lifts to a map $SL_1S^0 \xrightarrow{D} J$. Let $\text{Coker } J$ be the fiber of this last map.

At the prime 2 there are several possible definitions of J and consequently several possible definitions of $\text{Coker } J$. A perfectly reasonable approach is to set J to be the fiber of the map

$$(5.16) \quad BO^\otimes \xrightarrow{\psi^3/\psi^1} BO^\otimes.$$

However this introduces some homotopy groups in low degrees that are not in the image of D . To rectify this there are variations where one replaces one or both copies of BO by either its 1 or 2-connected cover. Rather than go through all the variations we note that all possible choices will yield the same definition of J after taking 1-connected covers. So we define J to be the 1-connected cover of the fiber of the map in (5.16). We then set $\text{Coker } J$ to be the fiber of the map $SL_1S^0\langle 1 \rangle \xrightarrow{D} J$.

It is a non-trivial fact that all spaces and maps in sight are infinite loop maps [May77, HS78]. We will follow tradition and denote their associated spectra with lower case letters.

Example 5.17. Let $X = \Sigma_+^\infty \text{Coker } J$ be the unreduced suspension spectrum of the infinite loop space $\text{Coker } J$ and let R be any E_∞ ring spectrum. The T -algebra spectral sequence computing the homotopy of $E_\infty(X, L_{K(2)}R)$ collapses at the E_2 page onto the 0-line. So in this case the edge homomorphism

$$\pi_0 E_\infty(X, L_{K(2)}R) \rightarrow H_\infty(X, L_{K(2)}R)$$

is an isomorphism. Moreover there is a homotopy equivalence of spaces

$$E_\infty(X, L_{K(2)}R) \simeq \Omega^\infty L_{K(2)}R.$$

These results follow from Example 5.10 and the following Theorem 5.18.

Theorem 5.18. [Kuhn-Noel] There is a $K(2)$ -equivalence of E_∞ ring spectra

$$TS^0 \simeq \Sigma_+^\infty \text{Coker } J$$

where T is the monad whose algebras are E_∞ ring spectra.

Proof. A consequence of the main result of [Kuh06, Thm. 2.21] is that for any spectrum X there is a natural map of E_∞ ring spectra

$$(5.19) \quad TX \rightarrow L_{K(2)}\Sigma_+^\infty \Omega^\infty X$$

which is an equivalence if $\pi_i X = 0$ for $i \leq 2$, torsion for $i = 3$, and $K(1)_* \Omega^\infty X$ is trivial.

First we consider the p -local case for an odd prime p . In this case the D -invariant

$$SL_1 S^0 \xrightarrow{D} J$$

is at least $2p - 1$ connected, hence $\text{Coker } J$ is at least $2p - 1 > 3$ -connected.

To see that $K(1)_* \text{Coker } J$ is trivial consider the defining fibration sequence of infinite loop spaces

$$(5.20) \quad \text{Coker } J \rightarrow SL_1 S^0 \xrightarrow{D} J.$$

By [HS78, Thm. 2.5] the D is a $K(1)$ -equivalence. Applying the $K(1)$ -Serre spectral sequence for this fibration, we see that the local coefficient system is trivial, the edge homomorphism is an isomorphism, and the spectral sequence collapses forcing $\text{Coker } J$ to be $K(1)$ -acyclic. Hence (5.19) is an equivalence for $X = \text{coker } j$.

Delooping (5.20) we obtain an exact triangle

$$\text{coker } j \rightarrow sl_1 S^0 \xrightarrow{d} j.$$

Since j is $K(2)$ -acyclic we have a $K(2)$ -equivalence $\text{coker } j \rightarrow sl_1 S^0$. There is also a homotopy equivalence $SL_1 S^0 \simeq QS_0^0$ between the 1 and 0 components of QS^0 . Although this is not a map of infinite loop spaces, applying the Bousfield-Kuhn functor ϕ_2 to this equivalence does yield an equivalence $L_{K(2)} sl_1 S^0 \simeq L_{K(2)} S^0 \langle 0 \rangle$.

Since Eilenberg-MacLane spectra are $K(n)$ -acyclic [RW80], the defining exact triangle for the 0-connected cover

$$S^0 \langle 0 \rangle \rightarrow S^0 \rightarrow H\mathbb{Z}$$

shows that $L_{K(2)} S^0 \langle 0 \rangle \simeq L_{K(2)} S^0$. Finally we use naturality of the spectral sequence

$$H_*(\Sigma_n; K(2)_*(X)^{\otimes_{K(2)_*} n}) \Rightarrow K(2)_*((E\Sigma_n)_+ \wedge_{\Sigma_n} X^n)$$

to see the functor T preserves $K(2)$ -equivalences.

Assembling these results, we obtain a zig-zag of equivalences of E_∞ ring spectra in the $K(2)$ -local category

$$TS^0 \leftarrow T(S^0 \langle 0 \rangle) \leftarrow Tsl_1 S^0 \leftarrow T\text{coker } j \rightarrow \Sigma_+^\infty \text{Coker } J.$$

At the prime 2 our defining fibration sequence is

$$\text{Coker } J \rightarrow SL_1 S^0 \langle 1 \rangle \xrightarrow{D} J.$$

Here D is 3-connected so $\text{Coker } J$ is sufficiently connected. Again the map D is a $K(1)$ -equivalence and consequently $\text{Coker } J$ is $K(1)$ -acyclic. The rest of the argument proceeds as before to obtain a zig-zag of $K(2)$ -local equivalences of E_∞ ring spectra

$$TS^0 \leftarrow T(S^0 \langle 1 \rangle) \leftarrow T(sl_1 S^0 \langle 1 \rangle) \leftarrow T\text{coker } j \rightarrow \Sigma_+^\infty \text{Coker } J.$$

□

Remark 5.21. This result can now be easily combined with the work of [Kas98, Str98] to determine the $K(2)$ and E_2 -cohomology of $\text{Coker } J$.

5.5. Computational lemmas. One of the key steps to obtaining a calculational description of the E_2 term is condition (b) from Theorem 5.6. That is we need to find a monad T_{alg} such that there is a natural isomorphism

$$\pi_* T \cong T_{\text{alg}} \pi_*.$$

In the examples below the category of T_{alg} -algebras will be equivalent to graded associative or commutative k -algebras which fit into the classical work [Qui69, Qui70] of Quillen guaranteeing the André-Quillen cohomology groups agree with the cotriple cohomology groups.

If R is a commutative S -algebra (see [EKMM97]) then we consider conditions (a) and (b) when T is a monad on R -module spectra whose category of algebras is equivalent to the category of A_∞ or E_∞ algebras in the category of R -modules.

In both of these examples we see that our monad takes the form

$$TM = \bigvee_{n \geq 0} K_n \otimes_{\Sigma_n} M^{\wedge R^n}.$$

In the E_∞ case K_n is equivariantly contractible while in the A_∞ case it is equivariantly weakly equivalent to Σ_n . To determine $\pi_* TM$ as a functor of $\pi_* M$ we will use a sequence of spectral sequence arguments that will require increasingly strong assumptions. These assumptions are clearly satisfied in the examples coming from Eilenberg-MacLane spectra in Section 5.3.

Lemma 5.22. If M and N are R -modules such that either $\pi_* M$ or $\pi_* N$ is flat as a $\pi_* R$ module then

$$\pi_*(M \wedge_R N) \cong \pi_* M \otimes_{\pi_* R} \pi_* N$$

Proof. The Tor spectral sequence of [EKMM97, IV.4.1] collapses. □

Lemma 5.23. If M and N are R -modules such that $\pi_* M$ is projective as a $\pi_* R$ module then

$$\pi_t(R\text{-Mod}(M, N)) \cong \pi_* R\text{-Mod}(\pi_* M, \pi_{*+t} N).$$

Proof. The Ext spectral sequence of [EKMM97, IV.4.1] collapses. □

From these lemmas we easily deduce the following proposition which shows that condition (a) from Theorem 5.6 is satisfied.

Proposition 5.24. If M and N are R -module spectra and $\pi_* R$ is an algebra over a graded field then

$$\pi_t(R\text{-Mod}(M, N)) \cong \pi_* R\text{-Mod}(\pi_* M, \pi_{*+t} N).$$

We can also chain together the above lemmas to see that condition (b) from Theorem 5.6 is satisfied in the case of A_∞ -algebras in R -module spectra where $\pi_* R$ is an algebra over a graded field:

Proposition 5.25. If X is an R -module spectrum, $\pi_* R$ is an algebra over a graded field, and T is the monad on R -module spectra whose category of algebras is the category of A_∞ algebras in R -module spectra then there is a natural isomorphism

$$\pi_* TM \cong T_{\text{alg}} \pi_* M := \bigoplus_{n \geq 0} (\pi_* M)^{\otimes_{\pi_* R} n}.$$

Here T_{alg} is the monad on $\pi_* R$ -modules whose algebras are the associative algebras in that category.

To see that condition (b) from Theorem 5.6 is satisfied in the case of rational E_∞ -algebras, we use Proposition 5.27, which is an immediate application of Lemma 5.26.

Lemma 5.26. If $E\Sigma_n$ is a contractible Σ_n -CW-complex and M is an R -module such that $n!$ is a unit in $\pi_0 R$ then

$$\pi_*(E\Sigma_n \otimes_{\Sigma_n} M^{\wedge R^n}) \cong \pi_*(M^{\wedge R^n})/\Sigma_n.$$

Proof. The homotopy orbit spectral sequence

$$H_s(\Sigma_n; \pi_t(M^{\wedge_R n})) \Rightarrow \pi_{s+t}((E\Sigma_n)_+ \wedge_{\Sigma_n} M^{\wedge_R n})$$

collapses by a standard transfer argument since $n!$ acts invertibly on the coefficients. \square

Proposition 5.27. If X is an R -module spectrum, π_*R is a graded field, π_0R is a field of characteristic 0, and T is the monad on R -module spectra whose category of algebras is the category of E_∞ algebras in R -module spectra then there is a natural isomorphism

$$\pi_*TM \cong T_{\text{alg}}\pi_*M := \bigoplus_{n \geq 0} (\pi_*M)^{\otimes_{\pi_*R} n} / \Sigma_n.$$

Here T_{alg} is the monad on π_*R -modules whose algebras are the commutative algebras in that category.

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